Surface Reconstruction and Regularization

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Before starting

What is the meaning of...

SURFACE RECONSTRUCTION
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Computer vision community

Find some 3D features from multiple images.
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SURFACE RECONSTRUCTION

Computer vision community
Find some 3D features from multiple images.

CAD Community
Find a surface which approximates/interpolates a given set of data points.
Why?

- A surface is generally more convenient to manipulate than a set of points.
- It allows to make some computations easier: geodesics, areas, etc.
- Global shape editing
1. Introduction

2. Surface Reconstruction

3. Choosing the Regularization Parameter

4. The L-Tangent Norm

5. Experimental Results

6. Conclusion
Range Data

- Centers: \((x_i, y_i) \in [0, 1]^2 \subset \mathbb{R}^2\)
- Elevation (altitude, depth): \(z_i \in \mathbb{R}\)
Range Data

Centers: 
\((x_i, y_i) \in [0, 1]^2 \subset \mathbb{R}^2\)

Elevation (altitude, depth):
\(z_i \in \mathbb{R}\)

Not as general as a full 3D point cloud...

But we don’t have to care about the \textit{parametrization problem}
Range Data

Range Data

Centers:
\((x_i, y_i) \in [0, 1]^2 \subset \mathbb{R}^2\)

Elevation (altitude, depth):
\(z_i \in \mathbb{R}\)

However, range data are useful in real applications
Range Data

- Centers: \((x_i, y_i) \in [0, 1]^2 \subset \mathbb{R}^2\)
- Elevation (altitude, depth): \(z_i \in \mathbb{R}\)
Surface Model

\[ f : [0, 1]^2 \rightarrow \mathbb{R} \]
\[ (x, y) \mapsto z = f(x, y; p) \]

- Model of known form controlled by a \textit{vector of parameters}:
  \[ p = [p_1 \ldots p_h] \in \mathbb{R}^h \]
Surface Model

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- Model of known form controlled by a vector of parameters: \( p = [p_1 \ldots p_h] \in \mathbb{R}^h \)
- Here, we mainly consider linear models, that is:

\[ \forall (x, y) \in [0, 1]^2, \quad \exists v_{x,y} \in \mathbb{R}^h \quad \text{t.q.} \quad f(x, y; p) = v_{x,y}^T p \]

- Note that the surface model \( f \) is linear with respect to \( p \) but not necessarily with \((x, y)\).
Examples of Surface Models

**Bézier Surface**

\[
f(x, y; \mathbf{p}) = \sum_{i=0}^{n} \sum_{j=0}^{m} p_{i(m+1)+j+1} B_{i,n}(x) B_{j,m}(y)
\]

**Polynomial Basis of Bézier**

\[
B_{i,n}(x) = \binom{n}{i} x^i (1 - x)^{n-i} \quad i = 0, \ldots, n \quad x \in [0, 1]
\]
Examples of Surface Models

Tensor-Product B-splines

\[ f(x, y; p) = \sum_{i=-a}^{b} \sum_{j=-c}^{d} p_{(i+a)(d+c+1)+j+c+1} N_{i,a+1}(x) N_{j,c+1}(y) \]

B-splines Basis

\[
\begin{align*}
N_{i,1}(x) &= 1 & \text{if } x \in [\lambda_i, \lambda_{i+1}] \\
N_{i,1}(x) &= 0 & \text{if } x \not\in [\lambda_i, \lambda_{i+1}] \\
N_{i,k+1}(x) &= \frac{x-\lambda_i}{\lambda_{i+k}-\lambda_i} N_{i,k}(x) + \frac{\lambda_{i+k+1}-x}{\lambda_{i+k+1}-\lambda_{i+1}} N_{i+1,k}(x) & \text{otherwise}
\end{align*}
\]
Examples of Surface Models

Radial Basis Functions

\[ f(x, y; p) = \sum_{i=1}^{h} p_i \rho \left( \sqrt{(x - x_i)^2 + (y - y_i)^2} \right) \]

Radial Basis

\[ \rho : \mathbb{R}_+ \longrightarrow \mathbb{R} \]

\[ r \quad \longmapsto \quad \rho(r) \]

\[ \rho(r) = r^2 \log(r) \quad \rho(r) = \exp(-r^2) \]
Consider a set of $n$ range data points:

$$\{(x_i, y_i) \leftrightarrow z_i \mid i = 1, \ldots, n\}$$
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Consider a given surface model:

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Consider a set of $n$ range data points:

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Consider a given surface model:

$$f : [0, 1]^2 \rightarrow \mathbb{R}^2$$

Objective: find the vector of parameters $p \in \mathbb{R}^h$ such that:

$$\begin{align*}
f(x_1, y_1; p) &\approx z_1 \\
\vdots &\vdots \\
f(x_n, y_n; p) &\approx z_n
\end{align*}$$
A Minimization Problem

This can be seen as a *minimization problem*

\[ p^* = \arg \min_{p \in \mathbb{R}^h} \mathcal{E}_d(p) \]

where \( \mathcal{E}_d \) is the *data term*, that is, a function that measures the closeness of the surface to the whole set of data points.
A Minimization Problem

The data term $E_d$ can be chosen in many ways.
A Minimization Problem

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The uniform error

$$E_d(p) = \max_{i=1,...,n} |f(x_i, y_i; p) - z_i|$$
The data term $\mathcal{E}_d$ can be chosen in many ways.

The uniform error

$$\mathcal{E}_d(p) = \max_{i=1,\ldots,n} |f(x_i, y_i; p) - z_i|$$

The mean squared residual (MSR)

$$\mathcal{E}_d(p) = \frac{1}{n} \sum_{i=1}^{n} (f(x_i, y_i; p) - z_i)^2$$
If we consider the MSR as the data term:

$$\mathcal{E}_d(p) = \frac{1}{n} \sum_{i=1}^{n} (f(x_i, y_i; p) - z_i)^2$$

The surface reconstruction problem can be seen as a *linear least squares (LLS) problem*:

$$p^* = \arg \min_{p \in \mathbb{R}^h} \frac{1}{n} \| Mp - z \|^2_2$$
An LLS problem is easy to solve by means of the *pseudo-inverse matrix*:

\[ p^* = M^\dagger z \]

where

\[ M^\dagger = \left( M^T M \right)^{-1} M^T \]

is the pseudo-inverse of \( M \).
However, this approach is not sufficient enough!
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To overcome these problems, we introduce a regularization term $\mathcal{E}_r$:

$$p^* = \arg \min_{p \in \mathbb{R}^h} \left( \mathcal{E}_d(p) + \frac{\lambda}{1 - \lambda} \mathcal{E}_r(p) \right) \quad \lambda \in [0, 1]$$
To overcome these problems, we introduce a *regularization term* $\mathcal{E}_r$:

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$\mathcal{E}_r : \mathbb{R}^h \rightarrow \mathbb{R}_+$ is a measure of the regularity of the surface.

$$
\mathcal{E}_r(p) = \int \int_{\Omega} \left( \frac{\partial^2 f(x,y;p)}{\partial x^2} \right)^2 + \left( \frac{\partial^2 f(x,y;p)}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 f(x,y;p)}{\partial y^2} \right)^2 \, dx \, dy
$$
To overcome these problems, we introduce a regularization term $\mathcal{E}_r$:

$$p^* = \arg\min_{p \in \mathbb{R}^h} \left( \mathcal{E}_d(p) + \frac{\lambda}{1 - \lambda} \mathcal{E}_r(p) \right), \quad \lambda \in [0, 1]$$

$\lambda$, the regularization parameter, controls the trade-off between the goodness of fit and the regularity of the surface.

$\lambda = 0.0002$  $\lambda = 0.005$  $\lambda = 0.99$
Example

\[
F . \text{Brunet Surface Reconstruction}
\]
As for the data term:

\[ \mathcal{E}_r(p) = \frac{1}{n} \sum_{i=1}^{n} (f(x_i, y_i; p) - z_i)^2 \]

\[ = \frac{1}{n} \|Mp - z\|_2^2 \]

It is possible to write in matrix form the regularization term (by discretization means):

\[ \mathcal{E}_r(p) \approx \|Rp\|_2^2 \]
The regularized reconstruction surface problem writes as:

$$p^* = \arg \min_{p \in \mathbb{R}^h} \left\| \begin{bmatrix} M \\ \frac{\lambda}{1-\lambda} R \end{bmatrix} p - \begin{bmatrix} z \\ 0 \end{bmatrix} \right\|^2_2$$

This LLS problem is solved by:

$$p^* = \left( M^T M + \left( \frac{\lambda}{1-\lambda} \right)^2 R^T R \right)^{-1} M^T z$$
Outline

1. Introduction
2. Surface Reconstruction
3. Choosing the Regularization Parameter
4. The L-Tangent Norm
5. Experimental Results
6. Conclusion
One of the challenges of surface reconstruction:

Choosing *automatically* the regularization parameter.
Choosing the Regularization Parameter

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Choosing *automatically* the regularization parameter.

- Lots of methods have been proposed: cross-validation, L-curve, etc.
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Choosing the Regularization Parameter

One of the challenges of surface reconstruction:
Choosing *automatically* the regularization parameter.

- Lots of methods have been proposed: cross-validation, L-curve, etc.
- These methods are usually expensive to compute
- Their optimization is sometimes difficult
Note

It doesn’t make sense to choose the optimal regularization parameter \( \lambda^* \) by including it in the initial minimization problem:

\[
(p^*, \lambda^*) = \arg \min_{p \in \mathbb{R}^h, \lambda \in [0,1]} \left( E_d(p) + \frac{\lambda}{1 - \lambda} E_r(p) \right)
\]
It doesn’t make sense to choose the optimal regularization parameter ($\lambda^*$) by including it in the initial minimization problem:

$$(p^*, \lambda^*) = \arg \min_{p \in \mathbb{R}^h, \lambda \in [0, 1]} \left( \mathcal{E}_d(p) + \frac{\lambda}{1 - \lambda} \mathcal{E}_r(p) \right)$$

Thus, $\lambda^*$ is chosen as the minimizer of an another criterion $V : [0, 1] \rightarrow \mathbb{R}$:

$$\lambda^* = \arg \min_{\lambda \in [0, 1]} V(\lambda)$$
It doesn’t make sense to choose the optimal regularization parameter \((\lambda^*)\) by including it in the initial minimization problem:

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Thus, \(\lambda^*\) is chosen as the minimizer of another criterion \(V : [0,1] \rightarrow \mathbb{R} :\)

\[
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\]

\[
p^* = \arg\min_{p \in \mathbb{R}^h} \left( \mathcal{E}_d(p) + \frac{\lambda^*}{1 - \lambda^*} \mathcal{E}_r(p) \right)
\]
One of the most used method to choose the regularization parameter: the Cross-Validation [Wahba, 1979].
The Cross-Validation

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What is a good reconstructed surface?
It’s a surface that *generalizes well.*
The Cross-Validation

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What is a good reconstructed surface?
It’s a surface that generalizes well.

The Cross-Validation aims to quantify the ability of a surface to generalize.
Given a regularization parameter $\lambda$
Principle of the Cross-Validation

- Given a regularization parameter $\lambda$
- Reconstruct the surface on the initial data where the $i$th point has been removed
Given a regularization parameter $\lambda$

Reconstruct the surface on the initial data where the $i$th point has been removed

Take the deviation between the reconstructed surface and the omitted point
Principle of the Cross-Validation

- Given a regularization parameter \( \lambda \)
- Reconstruct the surface on the initial data where the \( i \)th point has been removed
- Take the deviation between the reconstructed surface and the omitted point
- Repeat this for all the points in the initial dataset
Principle of the Cross-Validation

- Given a regularization parameter $\lambda$
- Reconstruct the surface on the initial data where the $i$th point has been removed
- Take the deviation between the reconstructed surface and the omitted point
- Repeat this for all the points in the initial dataset
- Finally, the Cross-Validation score (for the given $\lambda$) is defined as the mean of all the previously computed deviations
Principle of the Cross-Validation

More concisely:

\[ V(\lambda) = \frac{1}{n} \sum_{i=1}^{n} (f(x_i, y_i; p^{[i]}_\lambda) - z_i)^2 \]

where \( p^{[i]}_\lambda \) is the optimal vector of parameters computed on the initial dataset with the \( i \)th point removed.
Principle of the Cross-Validation

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It is almost impossible to use this formula directly!

The computation of

\[ \lambda^* = \arg \min_{\lambda \in [0,1]} V(\lambda) \]

is too expensive!
Cross-Validation

[Wahba, 1979]

\[ V(\lambda) = \frac{1}{n} \left\| \text{diag} \left( \frac{1}{1 - \text{diag}(H)} \right) (I - H)z \right\|_2^2 \]

where \( H \) is the \textit{hat matrix}:

\[ H = M \left( M^T M + \left( \frac{\lambda}{1 - \lambda} \right)^2 R^T R \right)^{-1} M^T \]

\[ p^* = \left( M^T M + \left( \frac{\lambda}{1 - \lambda} \right)^2 R^T R \right)^{-1} M^T z \]

\[ \hat{z} = Mp^* \]

\[ \Rightarrow \hat{z} = Hz \]
Example

The L-Tangent Norm

F. Brunet

Surface Reconstruction

Choosing the Regularization Parameter

Experimental Results
Cross-Validation

Good approach

- when the dataset is really small
- e.g. twodimensional cases

Its use is (too) expensive

- large matrices inversion
- loss of matrix sparsity
Cross-Validation

Difficult to optimize

- oscillations at small scales
- pathological cases

![Graph 1](Image)

![Graph 2](Image)
What are we searching for?

A good *compromise* between the goodness of fit and the surface regularity.

- The data term $E_d(p^*_\lambda)$ shouldn’t be too large.
- The regularization term $E_r(p^*_\lambda)$ shouldn’t be too large.
Ideas [Lawson et Hanson, 1974]

The data term and the regularization term are plotted against each other.

\[
\log \| R_{p\lambda} \|^2_2
\]

\[
\log \| M p_{\lambda}^* - z \|^2_2
\]
L-curve

\[ \rho(\lambda) = \log \| M p_\lambda^* - z \| \quad \eta(\lambda) = \log \| R p_\lambda^* \| \]

\[ \{ (\rho(\lambda), \eta(\lambda)) \in \mathbb{R}^2 \mid \lambda \in [0, 1[ \} \]

\[ \kappa(\lambda) = 2 \frac{\rho'(\lambda) \eta''(\lambda) - \rho''(\lambda) \eta(\lambda)}{(\rho'(\lambda)^2 + \eta'(\lambda)^2)^{3/2}} \]

\[ \lambda^* = \arg \max_{\lambda \in [0, 1[} \kappa(\lambda) \]
The L-curve approach is much faster than the Cross-Validation

- The computation of the criterion itself doesn’t involve matrix inversions.
- Use of the matrix sparsity.
The L-curve approach is much faster than the Cross-Validation:

- The computation of the criterion itself doesn’t involve matrix inversions.
- Use of the matrix sparsity.

However, the shape of the criterion is often *pathological*:

- Multiple maxima
Choosing the Regularization Parameter

The L-Tangent Norm

Experimental Results

L-curve

\[ \log \| R p_\lambda^* \|_2 \]

\[ \log \| M p_\lambda^* - z \|_2^2 \]

\[ \kappa(\lambda) \]
Our Approach: the L-Tangent Norm

- A novel approach to choose the regularization parameter
- Based on the L-curve
- This method is as fast as the L-curve criterion
- And it is easier to optimize
Our Approach: the L-Tangent Norm

Fact
The parametrization of an L-curves is not uniform.

Idea
Choose the regularization parameter so that the norm of the tangent to the L-curve is minimal.
The L-Tangent Norm criterion has usually only one minimum (excluding the one reached for values of $\lambda$ close to 1)
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It sometimes happens that there are two minima. In such cases, both minima make sense.
Properties

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\[ L(\lambda) \]

\[ L(\lambda) \]
Main Advantages of the Method

- Easy to minimize
- Fast to compute
- Still usable with non-linear models
Experimentations have been done using two different types of data:

- Synthetic data
- Real range images
Data

Synthetic data

F. Brunet
Data

Real range images
Timings

Single point evaluation

- Single point evaluation of the criterion
- Plots in function of the dataset size
- L-Tangent Norm vs. Cross-Validation
- Synthetic data

![Graphs showing computation time and ratio of cross-validation time by the L-tangent norm.](image-url)
Timings

Optimization process

- Time needed by the optimization process
- L-Tangent Norm vs. Cross-Validation
- Synthetic data

![Graph showing computation times for Cross-validation and L-tangent norm](image)
Reconstruction of a whole surface

- Computation time needed to reconstruct the whole surface
- L-tangent Norm vs. Cross-Validation
- Real range images

![Graph showing computation times for L-tangent norm and cross-validation.](image)
Comparison with Cross-Validation

- 200 randomly generated surfaces
- $\lambda^*_c$: reg. param. obtained using CV
- $\lambda^*_l$: reg. param. obtained using LTN

The L-Tangent Norm criterion seems to be an approximation of the Cross-Validation one!
Comparison with Cross-Validation

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Integral Relative Error

- 200 randomly generated surfaces
- Comparison of the surfaces reconstructed using LTN, CV and LC with the initial surface (ground truth)
- Comparison made using the Integral Relative Error (IRE)

Integral Relative Error

\[ e(f, g) = \frac{\int\int_{\Omega} |g(x, y) - f(x, y)| \, dx \, dy}{\max_{(x,y)\in\Omega} f(x, y) - \min_{(x,y)\in\Omega} f(x, y)} \]

(with \(f\) the original surface and \(g\) the reconstructed one)
Integral Relative Error

Not surprisingly, the IRE for the surfaces reconstructed using the LTN and the CV are close from each other. On the other hand, the reconstructed surfaces obtained with LC are not really good.
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On the other hand, the reconstructed surfaces obtained with LC are not really good.
Relative Error Map

- Evaluation of the reconstruction goodness with real range images
- The reconstruction is done from a subset of the initial range images
- The reconstructed surface is compared to the initial range image using the Relative Error Map (REM)

Relative Error Map (REM)

(Difference of depth maps normalized)

\[
\left\{ \begin{array}{c}
\frac{|f(x, y) - g(x, y)|}{\max_{u,v} f(u, v) - \min_{u,v} g(u, v)} \\
(x, y) \in \Omega
\end{array} \right.
\]
Relative Error Map

Relative Error Map for the surface reconstructed using the LTN

The overall error is small.
Relative Error Map for the surface reconstructed using the LTN

The overall error is small.
Comparison with the Cross-Validation

Difference of the REM for LTN and CV.
Comparison with the Cross-Validation

Difference of the REM for LTN and CV.

There is no significant difference between the surfaces reconstructed using the LTN and the ones reconstructed using the CV.
Conclusion

- A novel approach has been proposed to select the regularization weight.
- Reconstructed surfaces using the L-Tangent Norm are as good as the ones reconstructed with Cross-Validation.
- The computational cost is much better than with the Cross-Validation.
- The optimization process is easier than with L-Curve or Cross-Validation.
- This approach is potentially usable with non-linear models.