

3D Computer Vision Conics

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Based on Prof. Navab's lecture slides for 3D Computer Vision

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Conics

Curve described by 2nd-degree equation in the plane

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

or *homogenized* $x \mapsto \frac{x_1}{x_3}, y \mapsto \frac{x_2}{x_3}$

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

or in matrix form

$$\mathbf{x}^T \mathbf{C} \mathbf{x} = 0 \quad \text{with} \quad \mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

5DOF: ratio of 6 parameters $\{a:b:c:d:e:f\}$

Five Points Define a Conic

For each point the conic passes through

$$ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f = 0$$

or

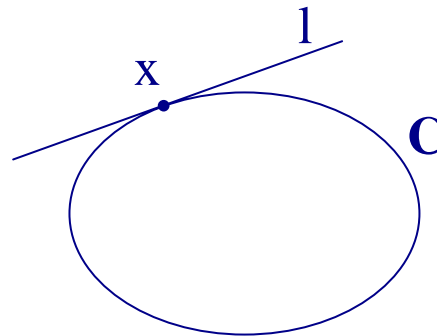
$$\begin{bmatrix} x_i^2 & x_iy_i & y_i^2 & x_i & y_i & 1 \end{bmatrix} \mathbf{c} = 0 \quad \mathbf{c} = [a, b, c, d, e, f]^T$$

stacking constraints yields

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \mathbf{c} = 0$$

Tangent Lines to Conics

The line l tangent to C at point x on C is given by $l=Cx$



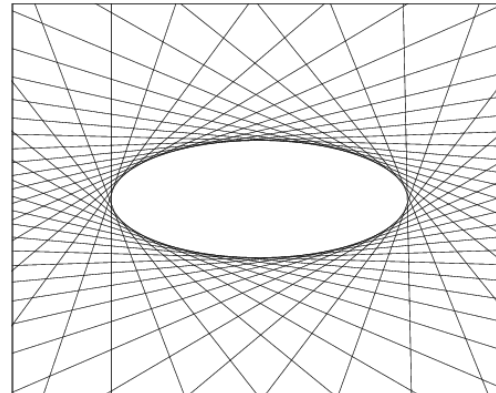
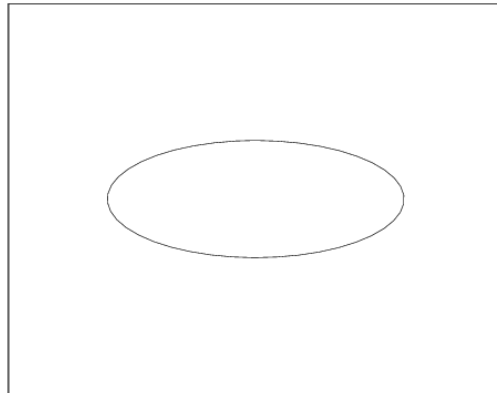
Dual Conics

A line tangent to the conic \mathbf{C} satisfies $\mathbf{1}^T \mathbf{C}^* \mathbf{1} = 0$

In case \mathbf{C} is full rank: $\mathbf{C}^* = \mathbf{C}^{-1}$

If not \mathbf{C}^* is the matrix of cofactors: $\mathbf{C}_{ij}^* = (-1)^{i+j} \det(\hat{\mathbf{C}}_{ij})$

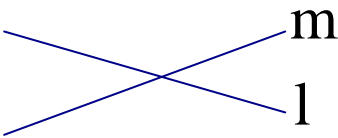
Dual conics = line conics = conic envelopes



Degenerate Conics

A conic is degenerate if matrix \mathbf{C} is not of full rank

e.g. two lines (rank 2)



$$\mathbf{C} = \mathbf{l}\mathbf{m}^T + \mathbf{m}\mathbf{l}^T$$

e.g. repeated line (rank 1)



$$\mathbf{C} = \mathbf{l}\mathbf{l}^T$$

Degenerate line conics: 2 points (rank 2), double point (rank1)

Note that for degenerate conics $(\mathbf{C}^*)^* \neq \mathbf{C}$

Transformation of Conics

Transformation of a conic C by the homography H

$$\mathbf{C}' = \mathbf{H}^{-T} \mathbf{C} \mathbf{H}^{-1}$$

Transformation of a dual conic C^* by the homography H

$$\mathbf{C}'^* = \mathbf{H} \mathbf{C}^* \mathbf{H}^T$$

The Circular Points

$$\mathbf{I} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \quad \mathbf{J} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

$$\mathbf{I}' = \mathbf{H}_s \mathbf{I} = \dots = \mathbf{I}$$

The circular points \mathbf{I} , \mathbf{J} are fixed points under the projective transformation \mathbf{H} if \mathbf{H} is a similarity

To be proofed in the exercises

Conic Dual to the Circular Points

$$\mathbf{C}_{\infty}^* = \mathbf{I}\mathbf{J}^T + \mathbf{J}\mathbf{I}^T$$

$$\mathbf{C}_{\infty}^* = \mathbf{H}_S \mathbf{C}_{\infty}^* \mathbf{H}_S^T$$

$$\mathbf{C}_{\infty}^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The dual conic \mathbf{C}_{∞}^* is fixed conic under the projective transformation \mathbf{H} if \mathbf{H} is a similarity

Note: \mathbf{C}_{∞}^* has 4DOF
 \mathbf{l}_{∞} is the nullvector

Measurement of Angles

Euclidean Space: $\mathbf{l} = (l_1, l_2, l_3)^\top$ $\mathbf{m} = (m_1, m_2, m_3)^\top$

$$\cos \theta = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}$$

We can use the conic dual to the circular points (\mathbf{C}_∞^*) in order to measure angles in projective space

Projective Space: $\cos \theta = \frac{\mathbf{l}^\top \mathbf{C}_\infty^* \mathbf{m}}{\sqrt{(\mathbf{l}^\top \mathbf{C}_\infty^* \mathbf{l})(\mathbf{m}^\top \mathbf{C}_\infty^* \mathbf{m})}}$

$$\mathbf{l}^\top \mathbf{C}_\infty^* \mathbf{m} = 0 \quad (\text{orthogonal})$$

Metric Properties from Images

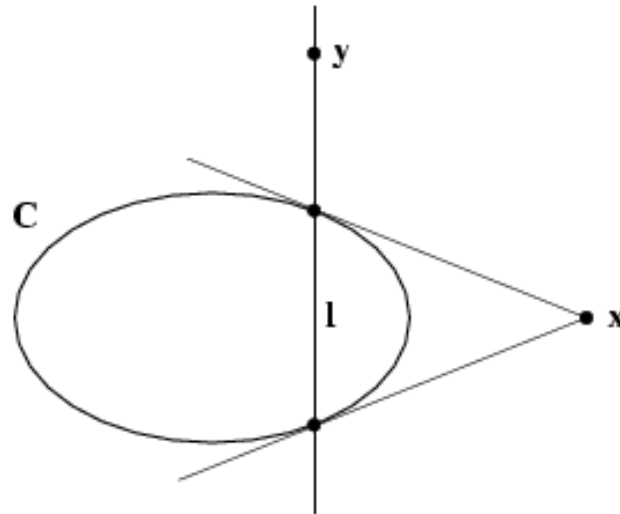
$$\begin{aligned}
 \mathbf{C}_{\infty}^* ' &= (\mathbf{H}_P \mathbf{H}_A \mathbf{H}_S) \mathbf{C}_{\infty}^* (\mathbf{H}_P \mathbf{H}_A \mathbf{H}_S)^{\top} \\
 &= (\mathbf{H}_P \mathbf{H}_A) \mathbf{H}_S \mathbf{C}_{\infty}^* \mathbf{H}_S^{\top} (\mathbf{H}_P \mathbf{H}_A)^{\top} \\
 &= (\mathbf{H}_P \mathbf{H}_A) \mathbf{C}_{\infty}^* (\mathbf{H}_P \mathbf{H}_A)^{\top} \\
 &= \begin{bmatrix} \mathbf{K} \mathbf{K}^{\top} & \mathbf{K}^{\top} \mathbf{V} \\ \mathbf{V}^{\top} \mathbf{K} & \mathbf{V}^{\top} \mathbf{V} \end{bmatrix}
 \end{aligned}$$

Rectifying transformation by applying the SVD

$$\mathbf{C}_{\infty}^* ' = \mathbf{U} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{U}^{\top} \quad \mathbf{H} = \mathbf{U}$$

Pole-Polar Relationship

The polar line $l = Cx$ of the point x with respect to the conic C intersects the conic in two points. The two lines tangent to C at these points intersect at x



Correlations and Conjugate Points

A *correlation* is an invertible mapping from points of P^2 to lines of P^2 . It is represented by a 3×3 non-singular matrix A as $l = Ax$

Conjugate points with respect to C
(on each others polar)

$$y^T C x = 0$$

Conjugate lines with respect to C^*
(through each others pole)

$$l^T C^* m = 0$$