

# Camera Calibration

Marco Feuerstein and Nassir Navab

based on a course given at UNC by Marc Pollefeys & the book "Multiple View Geometry" by Hartley & Zisserman

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chair for computer aided medical procedures

department of computer science | technische universität münchen

# Outline – Camera Model

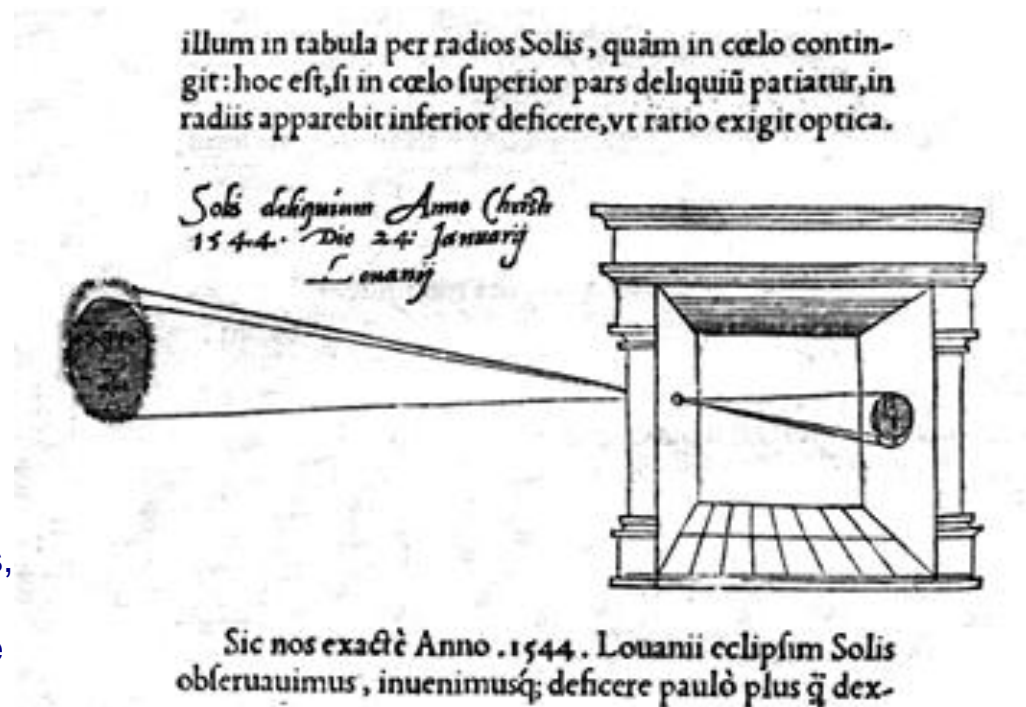
- History of Camera (Camera Obscura)
- Intensity Images
- Geometric Parameters of a Finite camera
- Projective Camera
  - Camera Center, Principal Plane, Axis Plane, Principal Point, Principal Ray
  - Decomposition of Camera Matrix

# History :: Camera Obscura

**Camera obscura** (||Camera obscura) [LL. camera chamber + L. obscurus, obscura, dark.] (Opt.)

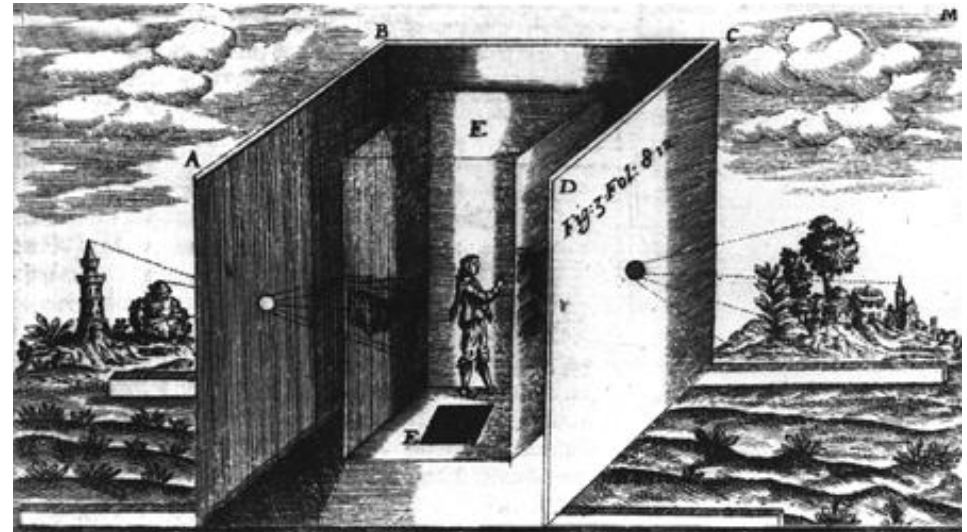
1. An apparatus in which the images of external objects, formed by a convex lens or a concave mirror, are thrown on a paper or other white surface placed in the focus of the lens or mirror within a darkened chamber, or box, so that the outlines may be traced.
2. (Photog.) An apparatus in which the image of an external object or objects is, by means of lenses, thrown upon a sensitized plate or surface placed at the back of an extensible darkened box or chamber variously modified; - commonly called simply the camera.

Websters Dictionary, 1913



Camera Obscura, Reinerus Gemma-Frisius, 1544, published in “*De Radio Astronomica et Geometrica*”, 1545 from Gernsheim, H., “The Origins of Photography”

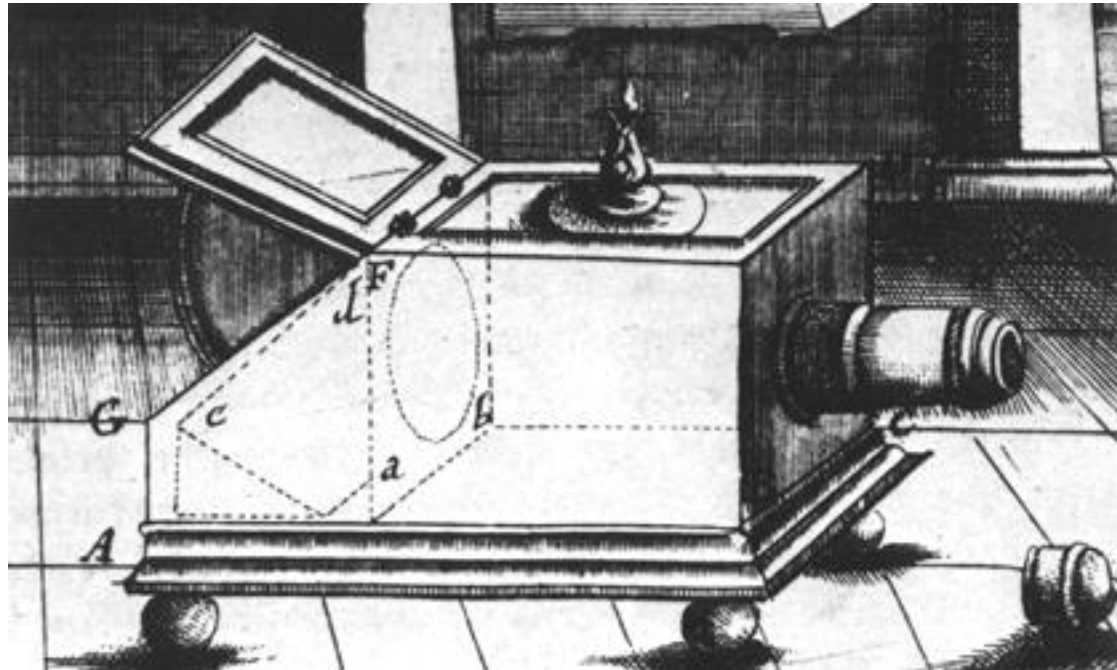
# History :: Camera Obscura



Camera Obscura, Athanasius Kircher, 1646  
from Gernsheim, H., *"The Origins of Photography"*

Portable 'Tent' Camera Obscura, Johannes Kepler, 1620  
from Gernsheim, H., *"The Origins of Photography"*

# History :: Camera Obscura



Reflex Camera Obscura, Johannes Zahn, 1685  
 published in Zahn, J., "*Oculus Artificialis*", (1685 - 1686)  
 from Gernsheim, H., "The Origins of Photography"

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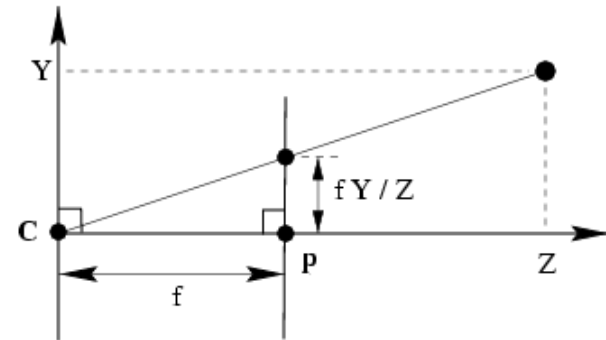
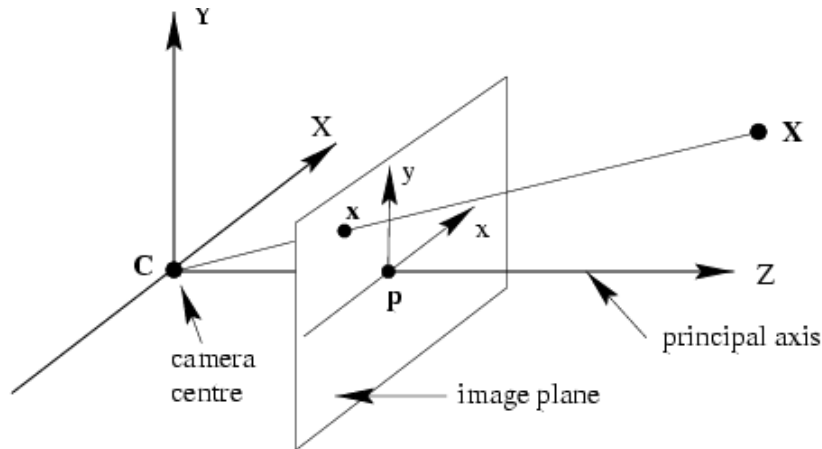
# Intensity Image

- photometric parameters
  - light energy reaching the sensors after being reflected from objects (not discussed in this course)
- **geometric parameters**
  - 2D position in the image on which a 3D point is projected (depending on camera geometry, discussed here)

# Outline – Camera Model

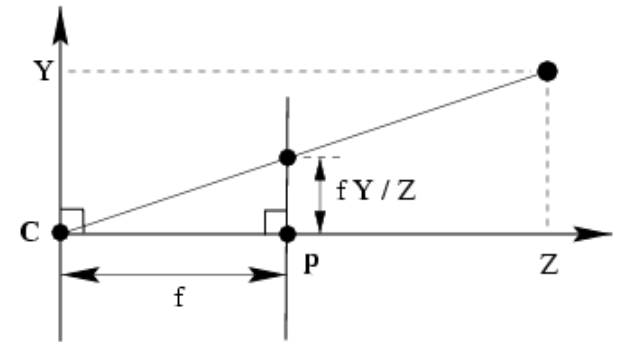
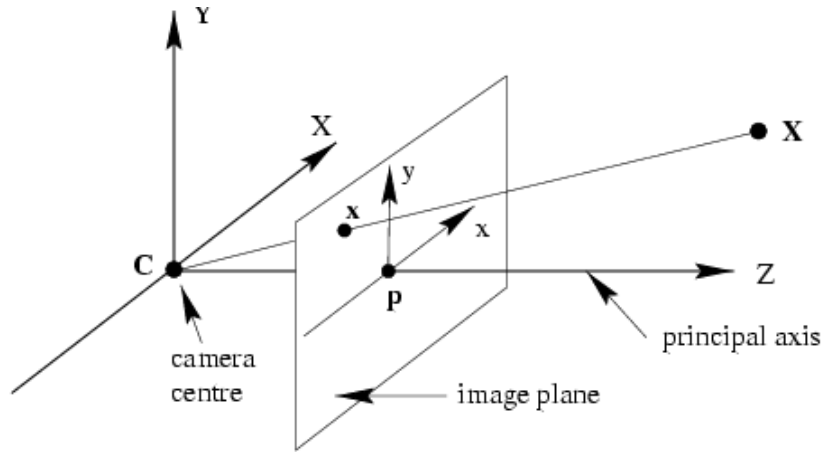
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# Pinhole Camera Model



- mapping between 3D world and 2D image  $(X, Y, Z)^T \mapsto (x, y)^T$
- central projection
- models are described in matrices with particular properties

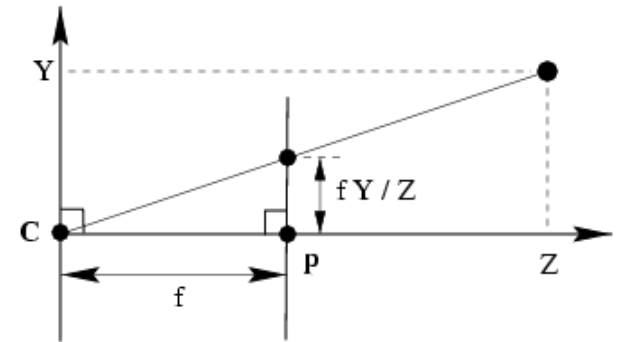
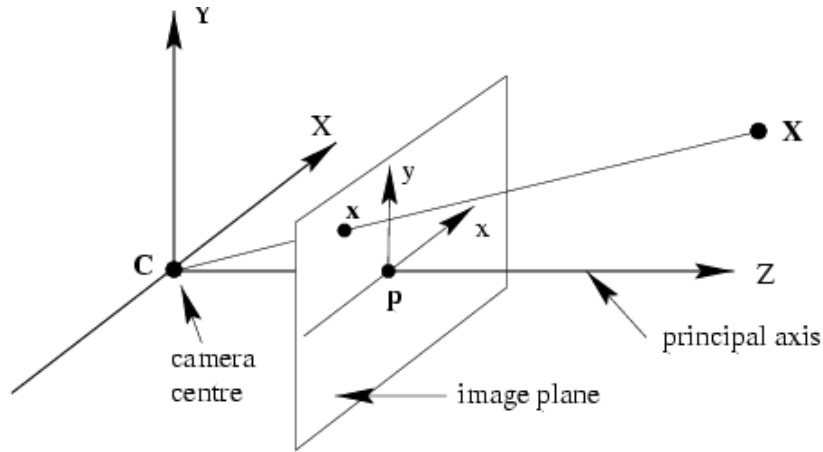
# Homogeneous Coordinates



$$(X, Y, Z, 1)^T \mapsto (x, y, z)^T$$

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

# Central Projection

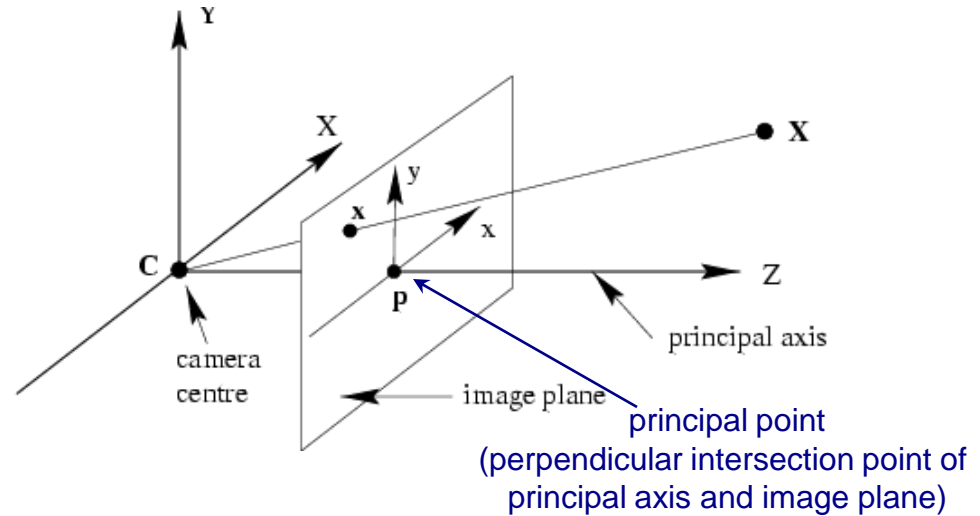
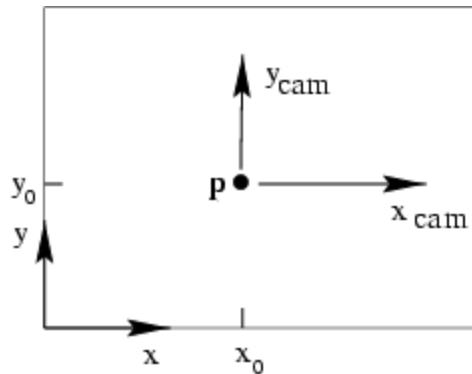


$$\begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & & \\ & f & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

$$\mathbf{P} = \text{diag}(f, f, 1) [\mathbf{I} \mid \mathbf{0}]$$

# Principal Point Offset



$$(X, Y, Z)^T \mapsto (fX/Z + p_x, fY/Z + p_y)^T$$

where  $(p_x, p_y)^T$

are the coordinates of the principal point

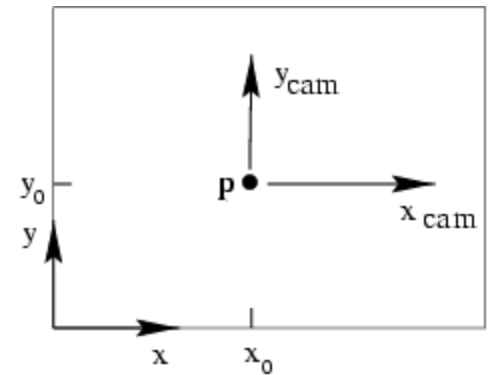
# Principal Point Offset

$$\begin{pmatrix} fX + Zp_x \\ fY + Zp_y \\ Z \end{pmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$\mathbf{x} = \mathbf{K}[\mathbf{I} \mid \mathbf{0}]\mathbf{X}$$

where  $\mathbf{K} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix}$

is called camera calibration matrix



# Camera Rotation and Translation

inhomogeneous coordinates

$$\tilde{\mathbf{X}}_{\text{cam}} = \mathbf{R}(\tilde{\mathbf{X}} - \tilde{\mathbf{C}})$$

where  $\tilde{\mathbf{X}} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$

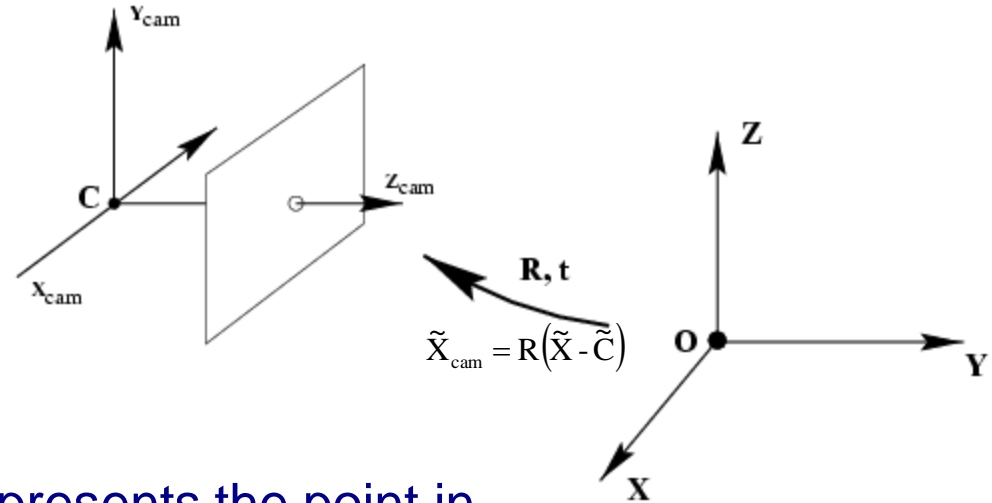
$$\tilde{\mathbf{X}}_{\text{cam}} = \begin{pmatrix} x_{\text{cam}} \\ y_{\text{cam}} \\ z_{\text{cam}} \end{pmatrix}$$

$$\tilde{\mathbf{C}}$$

represents the point in world coordinates

represents the same point in camera coordinates

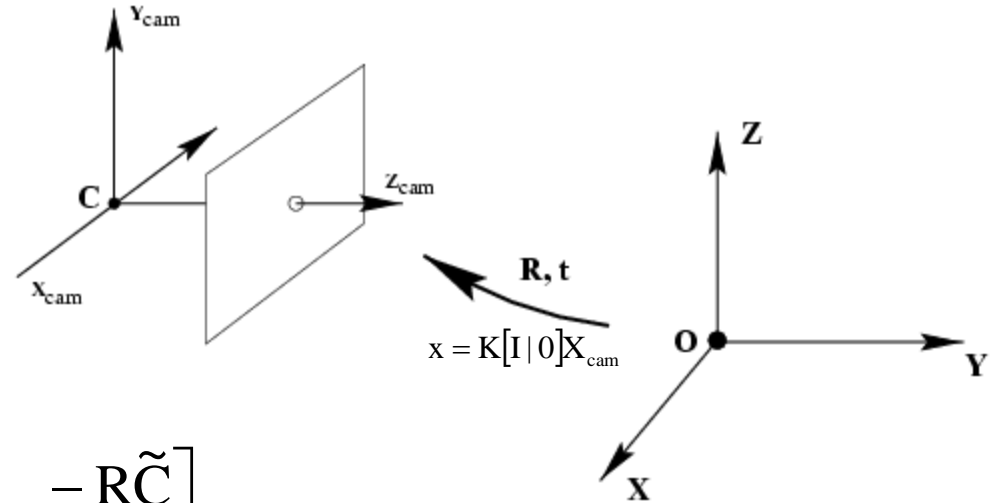
represents the coordinates of the camera origin in the world coordinate frame



# Camera Rotation and Translation

homogeneous coordinates

$$\mathbf{X}, \mathbf{X}_{cam} \in \mathbb{R}^4$$



$$\mathbf{X}_{cam} = \begin{bmatrix} \mathbf{R} & -\mathbf{R}\tilde{\mathbf{C}} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{Z} \\ 1 \end{pmatrix} = \begin{bmatrix} \mathbf{R} & -\mathbf{R}\tilde{\mathbf{C}} \\ 0 & 1 \end{bmatrix} \mathbf{X}$$

$$\mathbf{x} = \mathbf{K}[\mathbf{I} \mid 0]\mathbf{X}_{cam} \quad \text{projection to image plane from camera coordinates}$$

$$\mathbf{x} = \mathbf{K}\mathbf{R}[\mathbf{I} \mid -\tilde{\mathbf{C}}]\mathbf{X} \quad \text{projection to image plane from world coordinates}$$

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

# Extrinsic and Intrinsic Parameters

$$x = PX$$

where

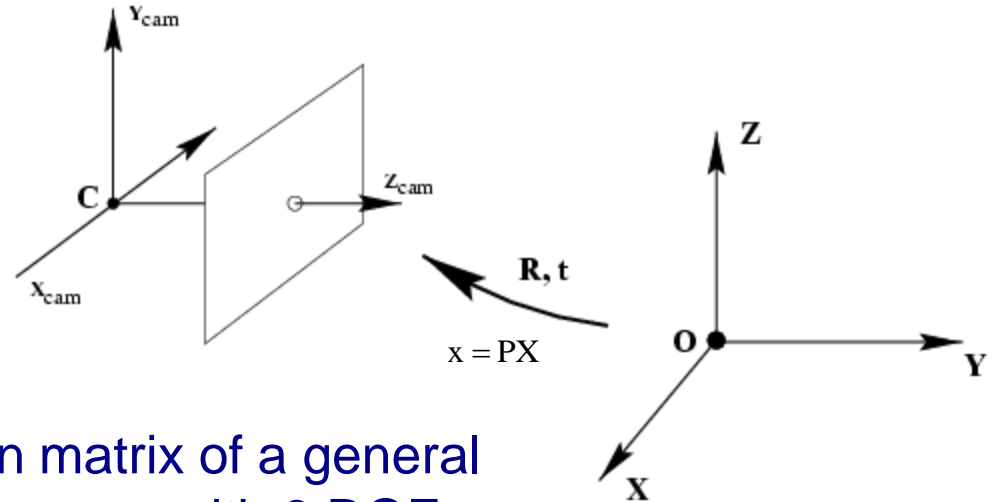
$$P = KR \left[ I \mid -\tilde{C} \right]$$

projection matrix of a general pinhole camera with 9 DOF

$$K = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$$

intrinsic camera parameters with 3 DOF

$R, \tilde{C}$  extrinsic camera parameters with each 3 DOF  
(camera orientation and position in world coordinates)



# Camera Rotation and Translation

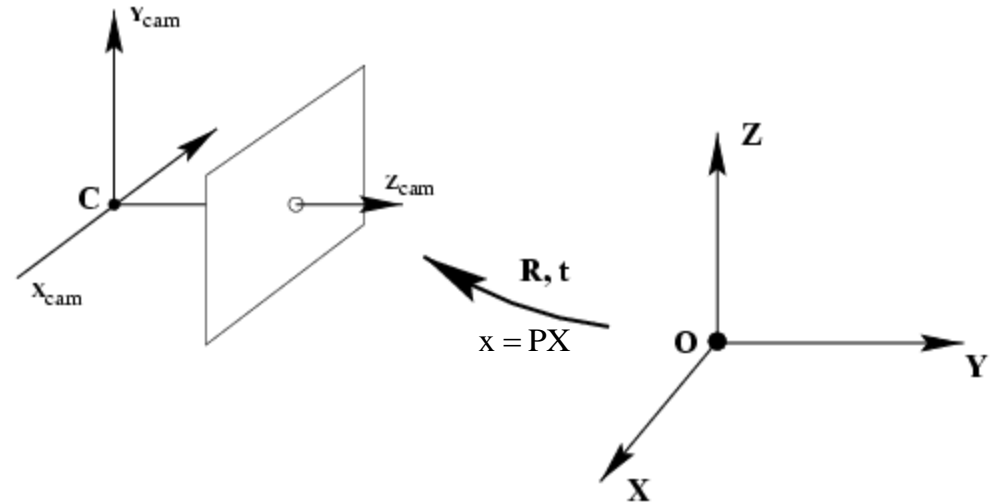
no explicit  
camera center

$$\tilde{\mathbf{X}}_{cam} = R\tilde{\mathbf{X}} + t$$

$$\begin{aligned} \mathbf{P} &= \mathbf{K}[\mathbf{R} \mid \mathbf{t}] \\ &= \mathbf{K}[\mathbf{R} \mid -R\tilde{\mathbf{C}}] \end{aligned}$$

where

$$t = -R\tilde{\mathbf{C}} \quad \text{from} \quad \mathbf{x} = \mathbf{KR}[\mathbf{I} \mid -\tilde{\mathbf{C}}]\mathbf{X}$$





# Skew Parameter

**S** skew parameter

$$K = \begin{bmatrix} \alpha_x & s & p_x \\ & \alpha_y & p_y \\ & & 1 \end{bmatrix} \quad 5 \text{ DOF}$$

$$P = KR \begin{bmatrix} I & | & -\tilde{C} \end{bmatrix} \quad \text{finite projective camera with 11 DOF}$$

# Finite Projective Camera :: Summary

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

$$\mathbf{P} = \begin{bmatrix} \alpha_x & s & p_x \\ & \alpha_y & p_y \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & -\mathbf{R}\tilde{\mathbf{C}} \end{bmatrix} \quad \begin{array}{l} \text{projection matrix} \\ 11 \text{ DOF } (5+3+3) \end{array}$$

$$= \underbrace{\mathbf{K}\mathbf{R}}_{\text{non-singular}} \begin{bmatrix} \mathbf{I} & | & -\tilde{\mathbf{C}} \end{bmatrix}$$

non-singular

# Finite Projective Camera :: Decomposition of P

$$\mathbf{M} = \mathbf{KR} \quad \text{non-singular } 3 \times 3 \text{ matrix (8 DOF)}$$

$$\mathbf{P} = \mathbf{KR} \left[ \mathbf{I} \mid -\tilde{\mathbf{C}} \right]$$

$$\mathbf{P} = \mathbf{M} \left[ \mathbf{I} \mid -\tilde{\mathbf{C}} \right]$$

decompose projection matrix P in K,R,C

$$\mathbf{P} = \left[ \mathbf{M} \mid \mathbf{p}_4 \right]$$

$$\left[ \mathbf{K}, \mathbf{R} \right] = \underbrace{RQ}_{\text{RQ matrix decomposition}}(\mathbf{M})$$

$$\tilde{\mathbf{C}} = -\mathbf{M}^{-1} \mathbf{p}_4$$

RQ matrix  
decomposition

## Finite Projective Camera :: Summary

$$\begin{aligned} \mathbf{P} &= \mathbf{KR} \left[ \mathbf{I} \mid -\tilde{\mathbf{C}} \right] && \text{where} && \mathbf{P} \in \mathbb{R}^{3 \times 4} \\ &= \mathbf{M} \left[ \mathbf{I} \mid -\tilde{\mathbf{C}} \right] \end{aligned}$$

- camera matrix  $\mathbf{P}$  is identical with the set of homogeneous  $3 \times 4$  matrices for which the left hand  $3 \times 3$  submatrix is non singular
- $\{\text{finite cameras}\} = \{\mathbf{P} \mid \det \mathbf{M} \neq 0\} = \{\mathbf{P} \mid \text{rank}(\mathbf{M}) = 3\}$
- If  $\text{rank}(\mathbf{P}) = 3$ , but  $\text{rank}(\mathbf{M}) < 3$ , then camera at infinity
- if  $\text{rank}(\mathbf{P}) < 3$  the matrix mapping will be a line or a point and not a plane (not a 2D image)

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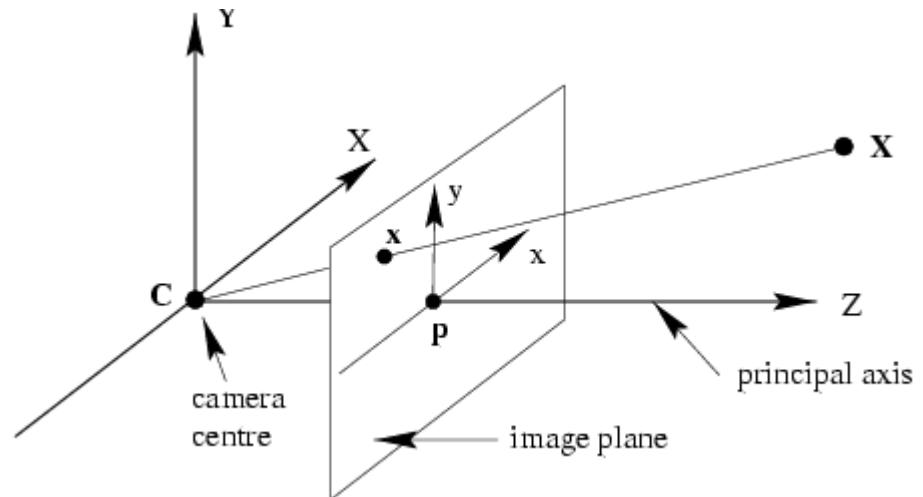
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# Camera Anatomy

- camera center
- column vectors
- principal plane
- axis plane
- principal point
- principal ray



# Camera Center

$$PC = 0 \quad P \text{ has a 1D null-space}$$

we will prove that the 4-vector  $C$  is the camera center

$$X(\lambda) = \lambda A + (1 - \lambda)C \quad \text{points on a line through } A \text{ and } C$$

$$x = PX = \lambda PA + (1 - \lambda)PC = \lambda PA \quad \text{since } PC = 0$$

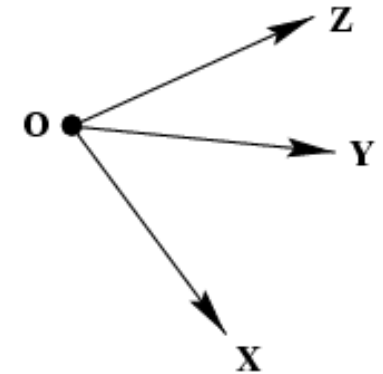
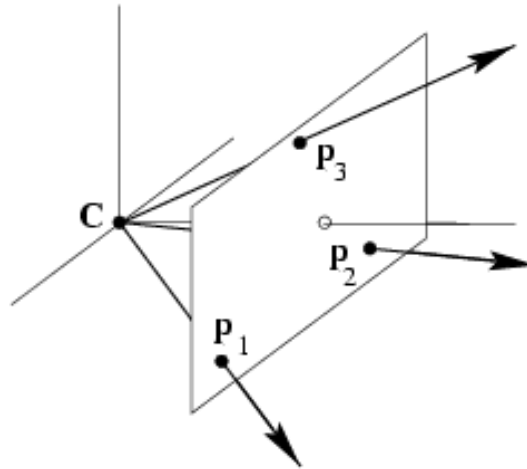
all 3D points on the line are mapped on the same 2D image point, and thus the line is a ray through the camera center

Finite cameras:

$$C = \begin{pmatrix} \tilde{C} \\ 1 \end{pmatrix} = \begin{pmatrix} -M^{-1}p_4 \\ 1 \end{pmatrix}$$

# Column Vectors

$$\mathbf{p}_i \in \mathbb{R}^3, i = 1, \dots, 4$$



column vectors are the image points, which project the axis directions (X,Y,Z) and the origin

example for the image

of the y-axis (vanishing pt)

$$[\mathbf{p}_2] = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3 \quad \mathbf{p}_4] \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T$$

$\mathbf{p}_4$  is the image of the world origin

# Row Vectors

$$\mathbf{p}^i \in \mathbb{R}^4, i = 1, \dots, 3$$

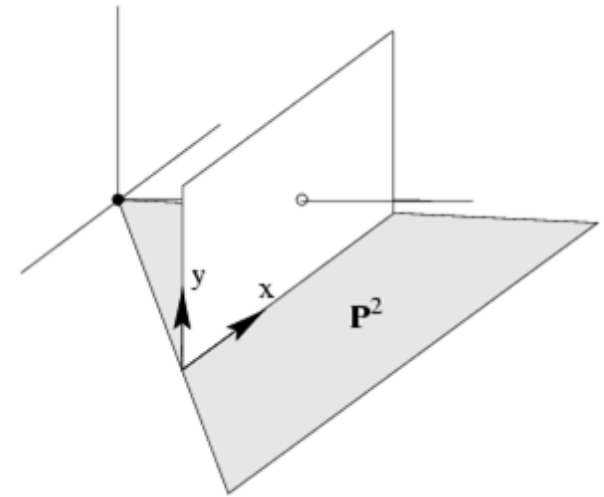
represent geometrically particular world planes

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} = \begin{bmatrix} \mathbf{p}^{1T} \\ \mathbf{p}^{2T} \\ \mathbf{p}^{3T} \end{bmatrix} \quad \text{row vectors}$$
$$= [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3 \quad \mathbf{p}_4] \quad \text{column vectors}$$

# Row Vectors of the Projection Matrix

$P^1$  is defined by the camera center and the line  $x=0$  on the image

$P^2$  is defined by the camera center and the line  $y=0$  on the image



Example  $P^2$

$$\begin{bmatrix} x \\ 0 \\ w \end{bmatrix} = \begin{bmatrix} \mathbf{p}^{1T} \\ \mathbf{p}^{2T} \\ \mathbf{p}^{3T} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

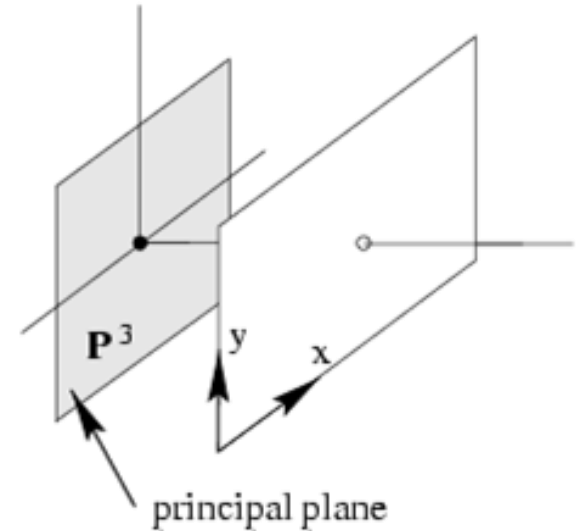
$$\mathbf{p}^{2T} \mathbf{X} = 0$$

$$\mathbf{p}^{2T} \mathbf{C} = 0$$

*respectively for  $P^1$*

# Principal Plane

plane through camera center and parallel to the image plane



$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{p}^{1T} \\ \mathbf{p}^{2T} \\ \mathbf{p}^{3T} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

points  $\mathbf{X}$  are imaged on the line at infinity

if  $\mathbf{X}$  is on the principle plane

$$\mathbf{p}^{3T} \mathbf{X} = 0$$

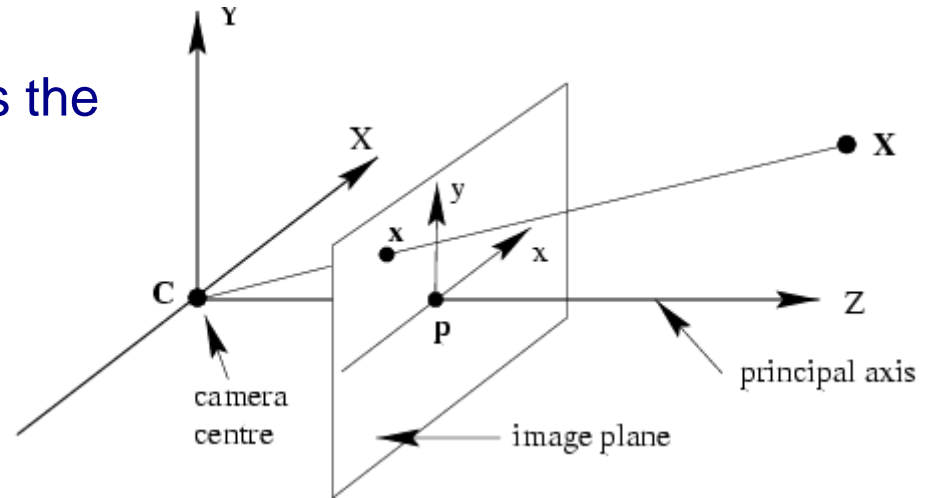
especially

$$\mathbf{p}^{3T} \mathbf{C} = 0$$

# Principle Point

line through camera center and perpendicular to principle plane is the principle axis

intersection of the principle axis with the image plane is the principle point



$\hat{\mathbf{P}}^3 = (p_{31} \ p_{32} \ p_{33} \ 0)^T$  normal direction to principal plane

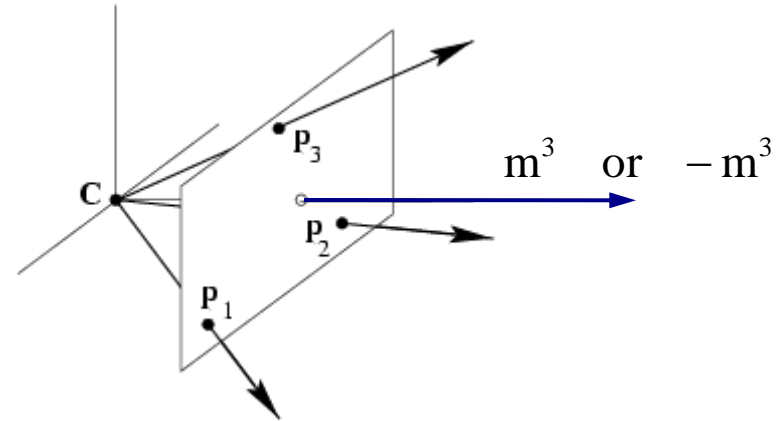
principle point

$$\mathbf{x}_0 = \mathbf{P}\hat{\mathbf{P}}^3 = \mathbf{M}\mathbf{m}^3 \quad \text{where} \quad \mathbf{P} = [\mathbf{M} \mid \mathbf{p}_4]$$

and  $\mathbf{m}^{3T}$  third row of  $\mathbf{M}$

# Principle Axis Vector

ambiguity that principle axis points towards the front of the camera (positive direction)



$$\mathbf{X} = \mathbf{P}_{\text{cam}} \mathbf{X}_{\text{cam}} = \mathbf{K} [\mathbf{I} \mid \mathbf{0}] \mathbf{X}_{\text{cam}}$$

$$\mathbf{v} = \det(\mathbf{M}) \mathbf{m}^3 = (0, 0, 1)^T$$

towards the front of the camera

$$\mathbf{P}_{\text{cam}} \mapsto k \mathbf{P}_{\text{cam}} \quad \mathbf{v} \mapsto k^4 \mathbf{v}$$

direction unaffected by scaling

$$\mathbf{P} = k \mathbf{K} \mathbf{R} [\mathbf{I} \mid -\tilde{\mathbf{C}}] = [\mathbf{M} \mid \mathbf{p}_4]$$

since  $\det(\mathbf{R}) > 0$

$$\mathbf{v} = \det(\mathbf{M}) \mathbf{m}^3$$

# Forward Projection

maps a point in space on the image plane  $\mathbb{P}^3 \rightarrow \mathbb{P}^2$

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

# Back-projection to Rays

points on the reconstructed ray

$$PC = 0 \quad \text{camera center } C$$

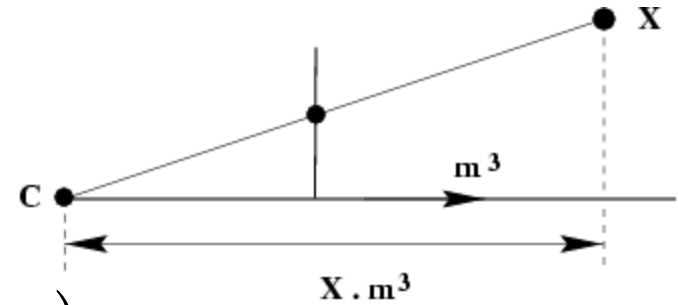
$$X = P^+ x \quad P^+ = P^T (PP^T)^{-1} \quad PP^+ = I$$

(pseudo-inverse)

ray is the line formed by those two points

$$X(\lambda) = P^+ x + \lambda C$$

# Depth of Points



$$w = P^{3T} X = P^{3T} (X - C) = m^{3T} (\tilde{X} - \tilde{C})$$

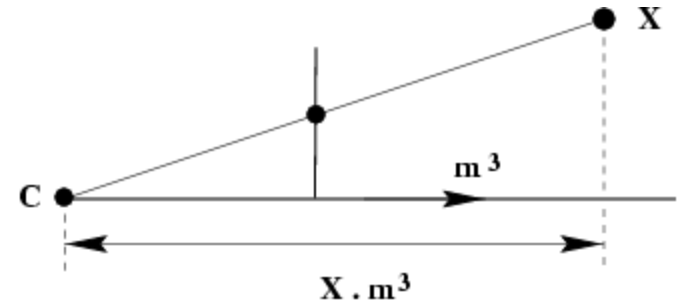
(PC=0)      (dot product)

If  $\det M > 0; \|m^3\| = 1$ ,  
 then  $m^3$  unit vector pointing in positive axis direction

Suppose  $P(X, Y, Z, T)^T = w(x, y, 1)^T$ . Then

$$\text{depth}(X; P) = \frac{\text{sign}(\det M) w}{T \|m^3\|}$$

# Depth of Points: examples



$$\text{depth}(X; P) = \frac{\text{sign}(\det M)w}{\|m^3\|}$$

$$P(X, Y, Z, 1)^T = w(x, y, 1)^T$$

$$\mathbf{x} = P\mathbf{X}$$

$$\mathbf{x} = P\mathbf{X}$$

$$P = [I \mid 0]$$

$$P = R [I \mid -\tilde{C}]$$

$$\text{depth}(X; P) = Z$$

$$\text{depth}(X; P) = R^3{}^T (\tilde{X} - \tilde{C})$$

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# Camera Matrix Decomposition

Finding the camera center  $C$

$$PC = 0$$

numerically: find right null-space by SVD of  $P$

$$P^{3 \times 4} = U^{3 \times 4} D^{4 \times 4} V^{4 \times 4 T} \rightarrow P^{3 \times 4} V^{4 \times 4} = U^{3 \times 4} D^{4 \times 4} \rightarrow PV^4 = 0$$

Algebraically:

$$X = \det([p_2, p_3, p_4]) \quad Y = -\det([p_1, p_3, p_4])$$

$$Z = \det([p_1, p_2, p_4]) \quad T = -\det([p_1, p_2, p_3])$$

where  $C = (X, Y, Z, T)^T$

# Camera Matrix Decomposition

Finding the camera center  $C$

$$PC = 0$$

Any plane  $\pi$  going through  $C$  will be a linear combination of the three planes defined by the rows of  $P$ . Therefore:

$$\begin{aligned} \det([\pi, P^T]) &= 0 \\ &= \pi_1 \det(P_{234}) - \pi_2 \det(P_{134}) + \pi_3 \det(P_{124}) - \pi_4 \det(P_{123}) \end{aligned}$$

$$X = \det([p_2, p_3, p_4]) \quad Y = -\det([p_1, p_3, p_4])$$

$$Z = \det([p_1, p_2, p_4]) \quad T = -\det([p_1, p_2, p_3])$$

where  $C = (X, Y, Z, T)^T$

# Camera Matrix Decomposition

Finding the camera orientation and internal parameters

$$P = [M \mid -M\tilde{C}] = K[R \mid -R\tilde{C}] \quad K = \begin{bmatrix} \alpha_x & s & y_0 \\ 0 & \alpha_y & x_0 \\ 0 & 0 & 1 \end{bmatrix}$$

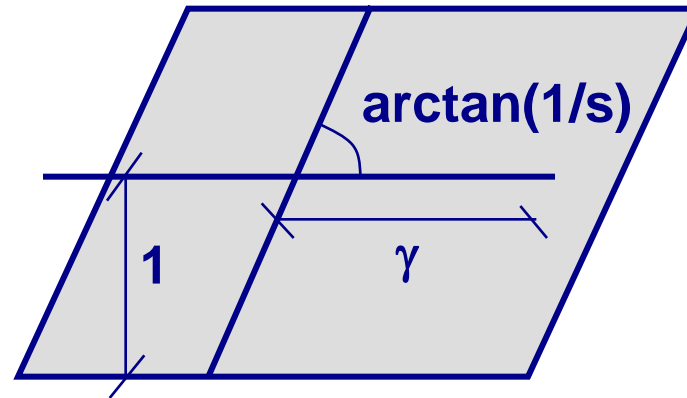
Decompose  $M = KR$  using RQ decomposition

$$\square = (\square Q \triangle R)^{-1} = \triangle R^{-1} \square Q^{-1}$$

Ambiguity removed by enforcing positive diagonal entries

# When is Skew Non-zero?

$$K = \begin{bmatrix} \alpha_x & s & p_x \\ & \alpha_y & p_y \\ & & 1 \end{bmatrix}$$



for CCD/CMOS, always  $s=0$

Image from image,  $s \neq 0$  possible  
(non coinciding principal axis)

resulting camera: **HP**

where H is a 3x3 homography

# Euclidean vs. Projective Spaces

general projective interpretation

$$P = \left[ \begin{array}{c} 3 \times 3 \text{ homography} \\ 4 \times 4 \text{ homography} \end{array} \right] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Meaningful decomposition in  $K, R, t$  requires  
Euclidean image and space

Camera center is still valid in projective space

Principal plane requires affine image and space

Principal ray requires affine image and Euclidean space

# Summary Camera Models

- photometric and radiometric properties of a camera
- geometric parameters of a finite camera
- projective cameras
  - camera anatomy (camera center, principle plane, principle point, and principle axis)
  - camera matrix decomposition (camera center, orientation, and intrinsic parameter)

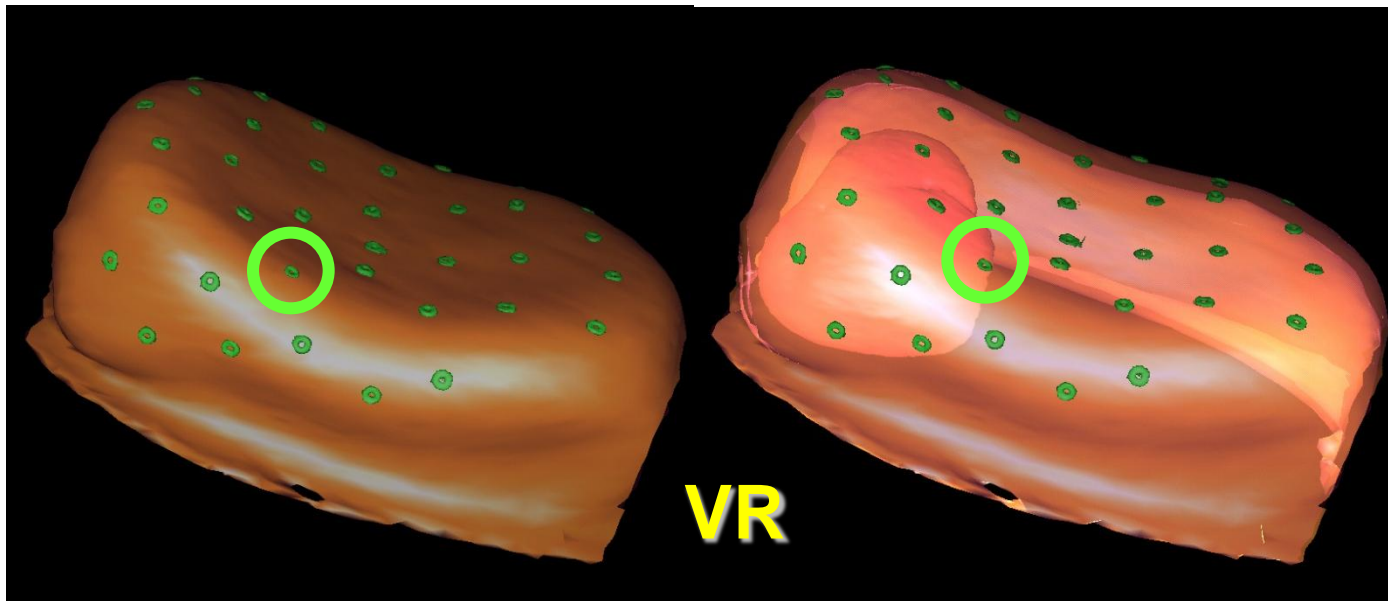
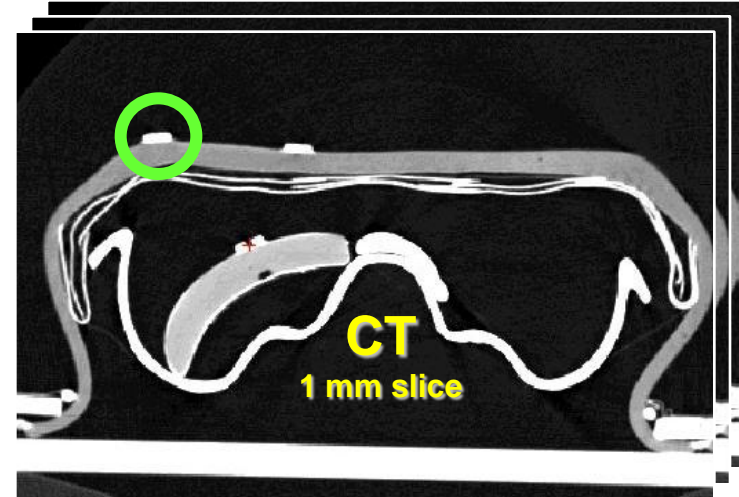
# Literature on Camera Models

- Chapter 6 in R. Hartley and A. Zisserman, “*Multiple View Geometry*”, 2<sup>nd</sup> edition, Cambridge University Press, 2003.
- Chapter 3 in O. Faugeras, “*Three-dimensional Computer Vision*”, MIT Press, 1993.
- Chapter 2 in E. Trucco and A. Verri, “*Introductory Techniques for 3-D Computer Vision*”, Prentice Hall, 1998.
- H. Gernsheim, “*The Origins of Photography*”, *Thames and Hudson*, 1982.
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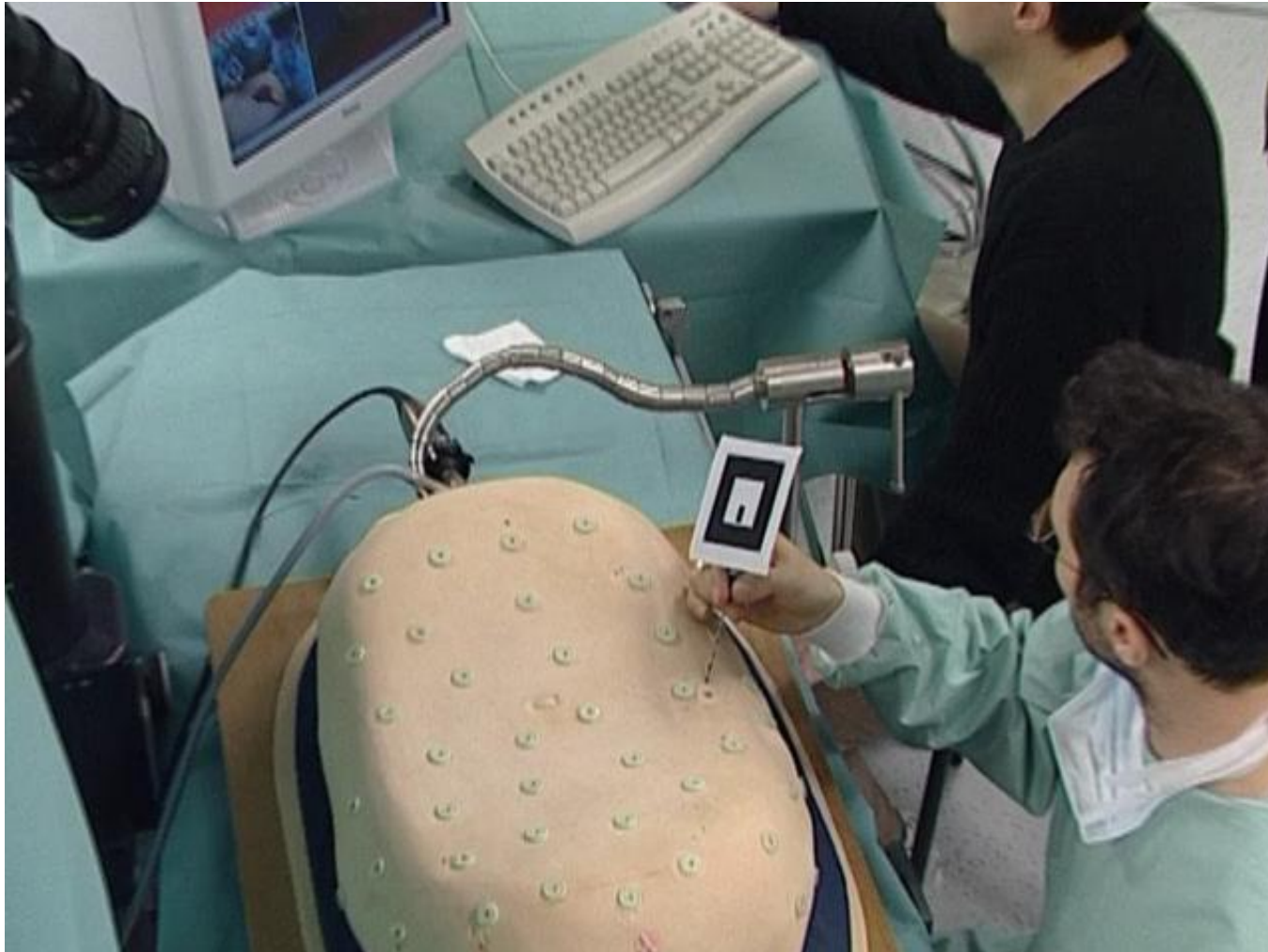
# Camera Calibration



# VR reconstruction of abdominal model (S. Nicolau)



# 3D/2D registration for Augmented Reality (S. Nicolau)



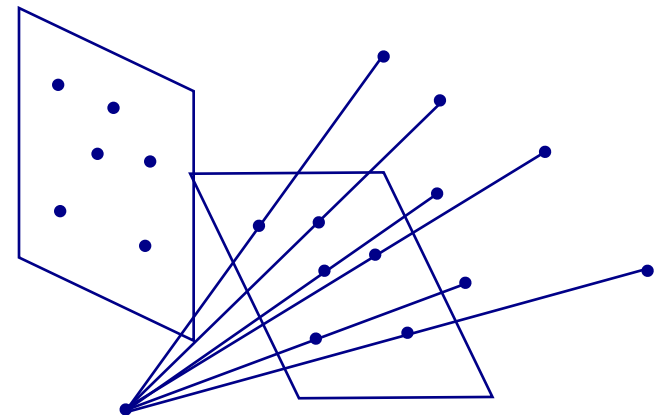
# Outline – Camera Calibration

- Basic Equations
- Geometric Error
- Radial and Tangential Distortion
- Exemplary Application of Calibrated Cameras

# Resectioning

- numerical methods for estimating the camera projection matrix from corresponding world and image entities
- simplest entities are the correspondence between a 3D point  $\mathbf{X}$  and its image  $\mathbf{x}$   $\mathbf{X}_i \leftrightarrow \mathbf{x}_i$
- $\mathbf{P}$  will be determined for sufficient many corresponding points (or other entities such as lines)

$\mathbf{P}$  ?



# Basic Equations

$$\mathbf{x}_i = \mathbf{P}\mathbf{X}_i \quad \text{find } \mathbf{P}$$

$$[\mathbf{x}_i]_{\mathbf{x}} \mathbf{P}\mathbf{X}_i = 0$$

$$\begin{bmatrix} \mathbf{0}^\top & -w_i \mathbf{X}_i^\top & y_i \mathbf{X}_i^\top \\ w_i \mathbf{X}_i^\top & \mathbf{0}^\top & -x_i \mathbf{X}_i^\top \\ -y_i \mathbf{X}_i^\top & x_i \mathbf{X}_i^\top & \mathbf{0}^\top \end{bmatrix} \begin{pmatrix} \mathbf{P}^1 \\ \mathbf{P}^2 \\ \mathbf{P}^3 \end{pmatrix} = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{0}^\top & -w_i \mathbf{X}_i^\top & y_i \mathbf{X}_i^\top \\ w_i \mathbf{X}_i^\top & \mathbf{0}^\top & -x_i \mathbf{X}_i^\top \end{bmatrix} \begin{pmatrix} \mathbf{P}^1 \\ \mathbf{P}^2 \\ \mathbf{P}^3 \end{pmatrix} = \mathbf{0}$$

$$\mathbf{A}\mathbf{p} = \mathbf{0}$$

# Basic Equations

$$Ap = 0 \quad \text{minimal solution}$$

P has 11 dof, 2 independent eq./points  
 $\Rightarrow 5\frac{1}{2}$  correspondences needed (say 6)

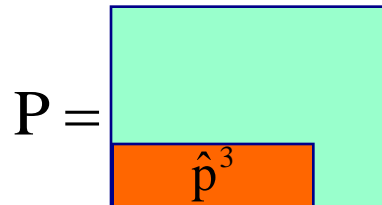
## Over-determined solution

$n \geq 6$  points

minimize  $\|Ap\|$  subject to constraint

$$\|p\| = 1$$

$$\|\hat{p}^3\| = 1$$



# DLT algorithm

## Objective

Given  $n \geq 6$  3D to 2D point correspondences  $\{X_i \leftrightarrow x_i\}$ , determine the Projection matrix  $P$  such that  $x_i = PX_i$

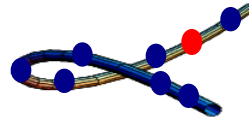
## Algorithm

- (i) For each correspondence  $X_i \leftrightarrow x_i$  compute  $A_i$ . Usually only two first rows needed.
- (ii) Assemble  $n$   $2 \times 12$  matrices  $A_i$  into a single  $2n \times 12$  matrix  $A$
- (iii) Obtain SVD of  $A$  (i.e.  $U \Sigma V^T$ ). Solution for  $p$  is last column of  $V$
- (iv) Determine  $P$  from  $p$

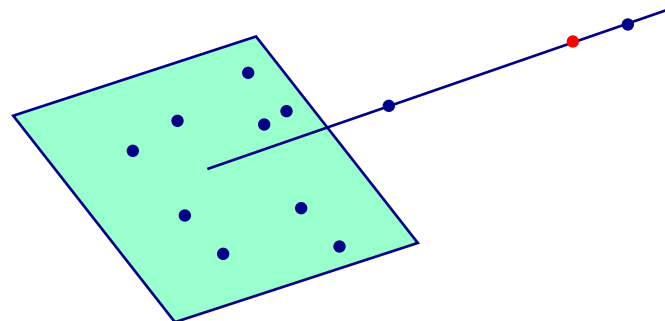
# Degenerate Configurations

More complicated than 2D case

- (i) Camera and points on a twisted cubic



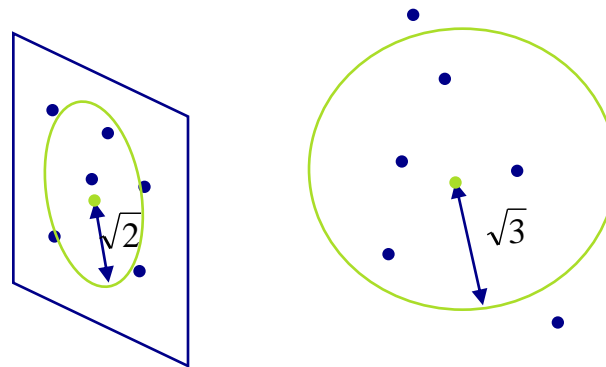
- (ii) Points lie on plane or single line passing through projection center



# Data Normalization

Less obvious

(i) Simple, as before

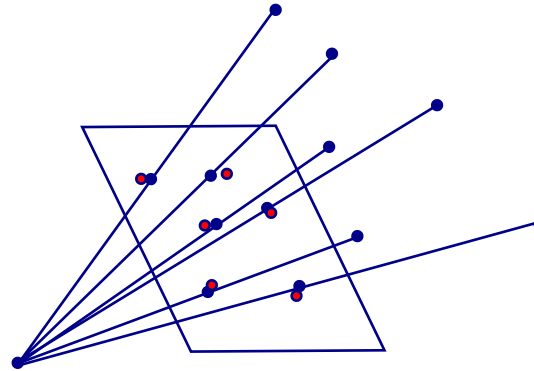


(ii) Anisotropic scaling

# Outline – Camera Calibration

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# Geometric Error



$$\sum_i d(\mathbf{x}_i, \hat{\mathbf{x}}_i)^2$$

$$\min_P \sum_i d(\mathbf{x}_i, P\mathbf{X}_i)^2$$

# Gold Standard Algorithm

for the estimation of P from world to image correspondences  
(no error in world points)

## Objective

Given  $n \geq 6$  3D to 2D point correspondences  $\{X_i \leftrightarrow x_i\}$ ,  
determine the Maximum Likelihood Estimation of P

## Algorithm

### (i) **Linear solution:**

(a) Normalization:  $\tilde{X}_i = UX_i$   $\tilde{x}_i = Tx_i$

(b) DLT:

(ii) **Minimization of geometric error:** using the linear estimate as a starting point minimize the geometric error:

$$\min_P \sum_i d(\tilde{x}_i, \tilde{P}\tilde{X}_i)^2$$

(iii) **Denormalization:**  $P = T^{-1}\tilde{P}U$

# Calibration Example

- (i) Canny edge detection
- (ii) (Straight) line fitting to the detected edges
- (iii) Intersecting the lines to obtain the images corners

typical precision  $< 1/10$

(HZ rule of thumb:  $5n$  constraints for  $n$  unknowns, i.e. at least 28 points)



	$f_y$	$f_x/f_y$	skew	$x_0$	$y_0$	residual
linear	1673.3	1.0063	1.39	379.96	305.78	0.365
iterative	1675.5	1.0063	1.43	379.79	305.25	0.364

# Algebraic Error

## Errors in the world

$$\sum_i d(\mathbf{X}_i, \hat{\mathbf{X}}_i)^2 \quad \mathbf{x}_i = \mathbf{P}\hat{\mathbf{X}}_i$$

## Errors in the image and in the world

$$\sum_{i=1}^n d_{\text{Mah}}(\mathbf{x}_i, \mathbf{P}\hat{\mathbf{X}}_i)^2 + d_{\text{Mah}}(\mathbf{X}_i, \hat{\mathbf{X}}_i)^2$$

# Geometric Interpretation of Algebraic Error

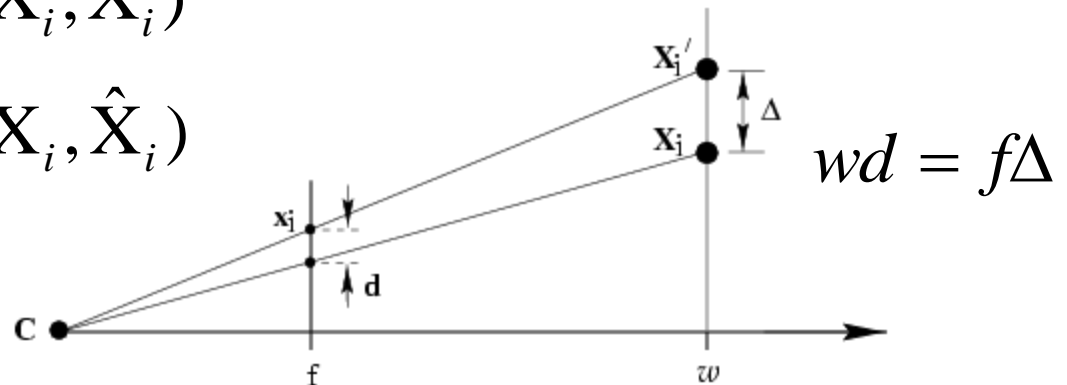
$$\sum_i (\hat{w}_i d(\mathbf{x}_i, \hat{\mathbf{x}}_i))^2$$

$$\hat{w}_i(\hat{x}_i, \hat{y}_i, 1) = \mathbf{P}\mathbf{X}_i \quad \hat{w}_i = \pm \|\hat{\mathbf{p}}^3\| \text{depth}(\mathbf{X}; \mathbf{P})$$

therefore, if  $\|\hat{\mathbf{p}}^3\| = 1$  then

$$\hat{w}_i d(\mathbf{x}_i, \hat{\mathbf{x}}_i) = f d(\mathbf{X}_i, \mathbf{X}'_i)$$

$$\hat{w}_i d(\mathbf{x}_i, \hat{\mathbf{x}}_i) \sim f d(\mathbf{X}_i, \hat{\mathbf{X}}_i)$$



note invariance to 2D and 3D similarities given proper normalization

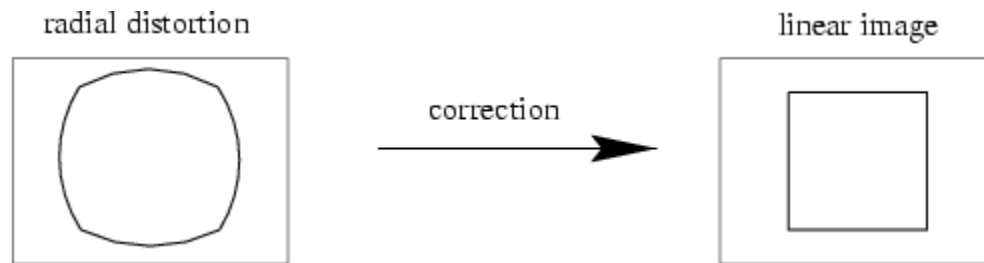
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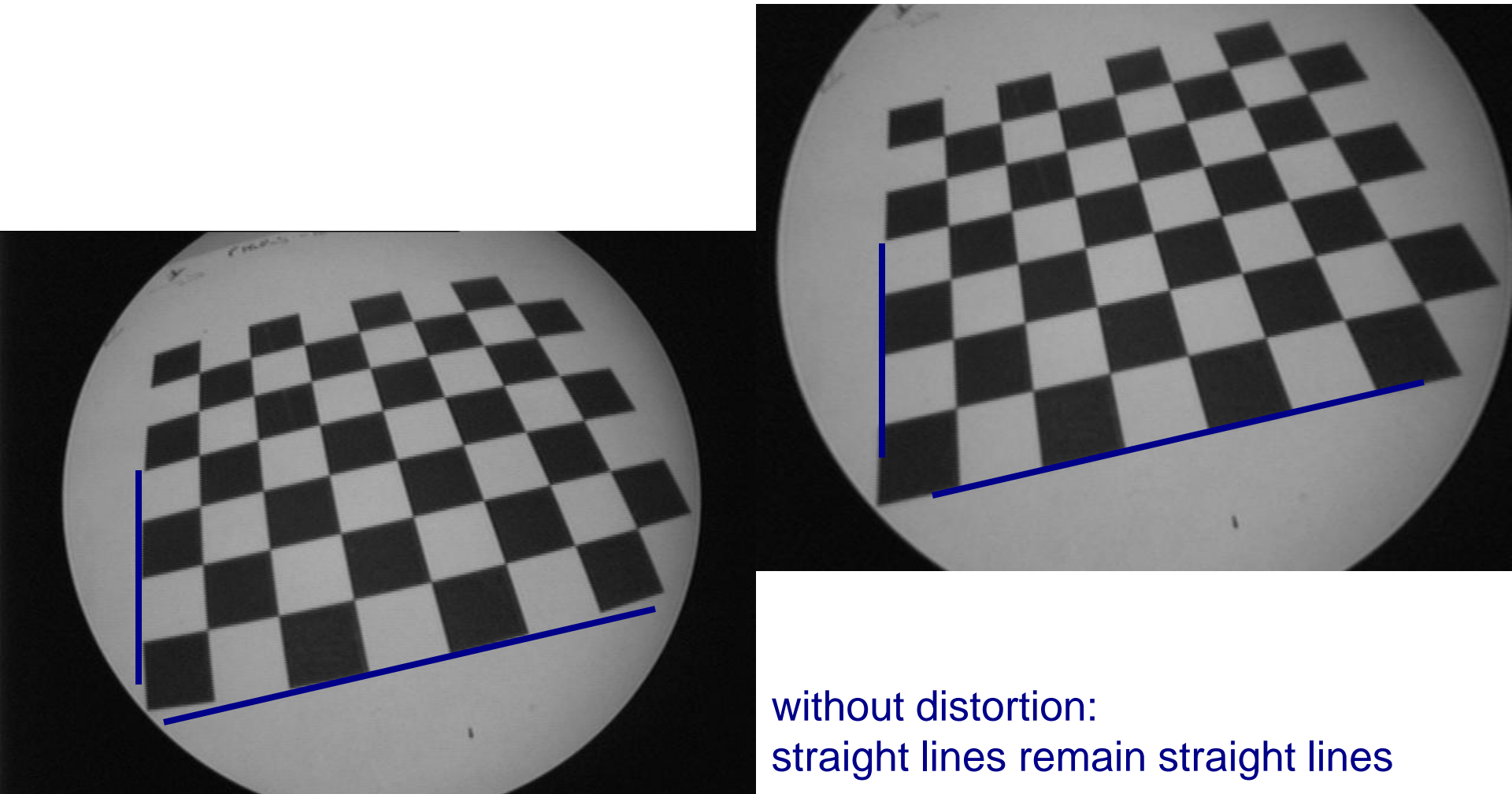
# Radial Distortion



short and long focal length

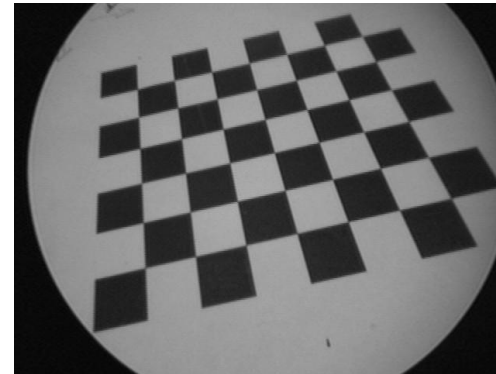
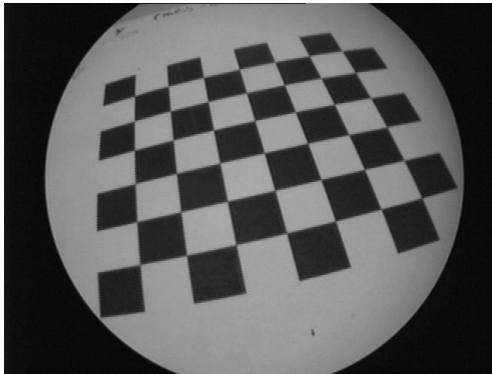


# Distorted and Undistorted Image



without distortion:  
straight lines remain straight lines

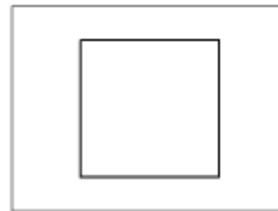
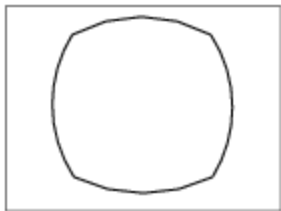
# Radial Distortion



radial distortion

linear image

correction



$$(\tilde{x}, \tilde{y}, 1)^\top = [\mathbf{I} \mid \mathbf{0}] \mathbf{X}_{\text{cam}}$$

$$\begin{pmatrix} x_d \\ y_d \end{pmatrix} = L(\tilde{r}) \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

# Correction of Radial and Tangential Distortion

Let  $x_n$  be the normalized (pinhole) image projection:

$$\mathbf{x}_n = \begin{bmatrix} X_c / Z_c \\ Y_c / Z_c \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \quad r^2 = x^2 + y^2$$

Choice of the distortion function, radial distortion center, and tangential distortion vector

$$\mathbf{x}_d = \begin{bmatrix} x_d(1) \\ x_d(2) \end{bmatrix} = \left( 1 + kc(1)r^2 + kc(2)r^4 + kc(5)r^6 \right) \mathbf{x}_n + \mathbf{dx}$$

$$\mathbf{dx} = \begin{bmatrix} 2 kc(3) x y + kc(4) (r^2 + 2x^2) \\ kc(3) (r^2 + 2y^2) + 2 kc(4) x y \end{bmatrix}$$

Computing the parameters of the distortion function

- (i) Minimize with additional unknowns
- (ii) Straighten lines
- (iii) ...

# Outline – Camera Calibration

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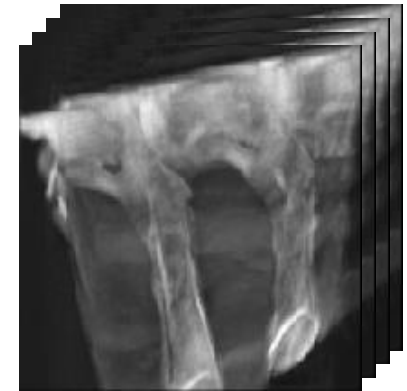
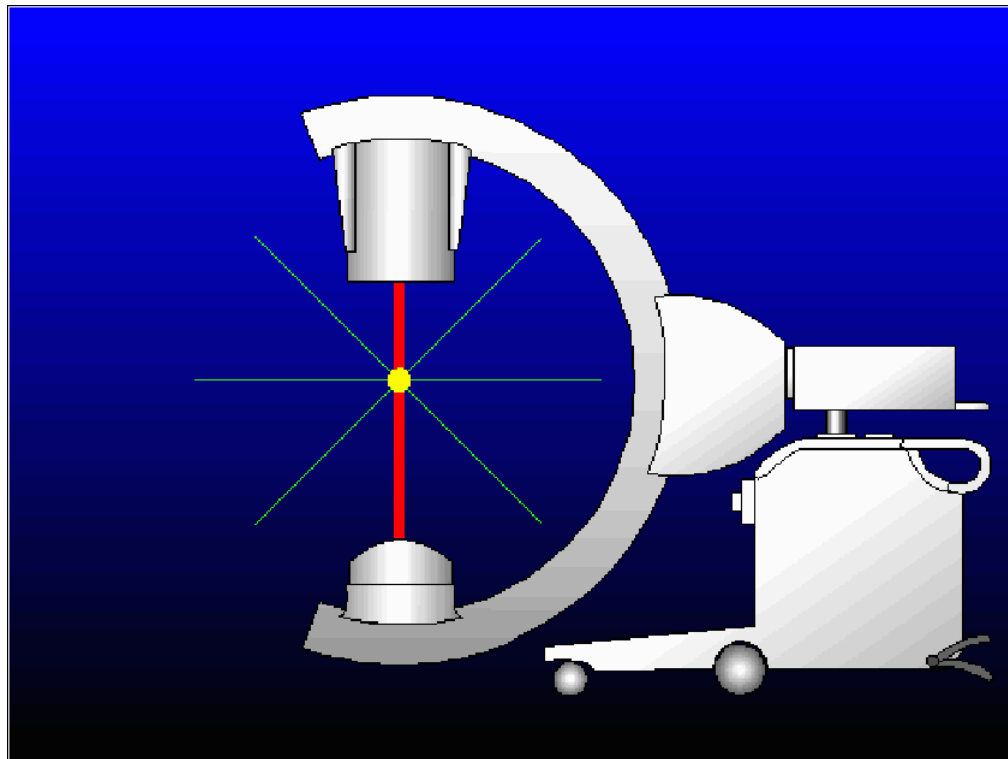
# Exemplary Application :: Intra-operative 3D Imaging



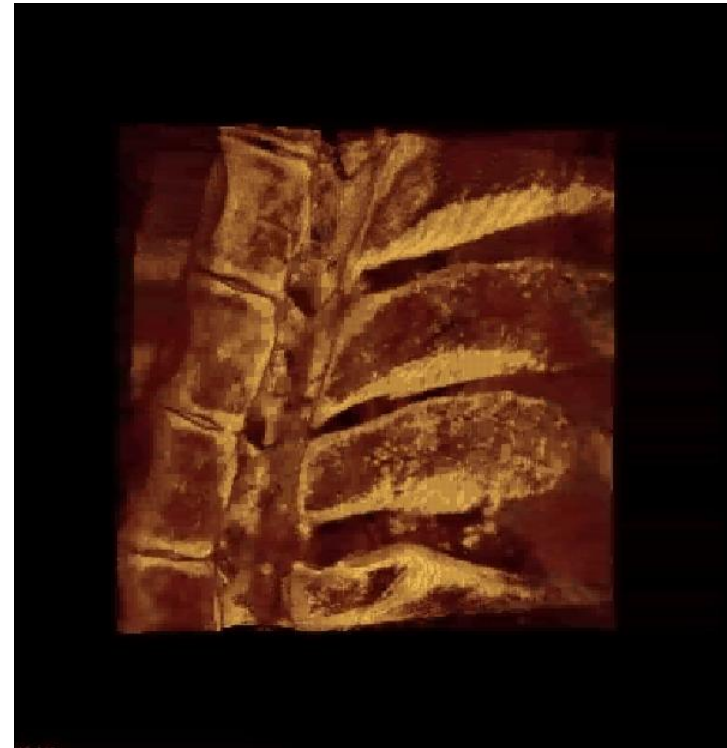
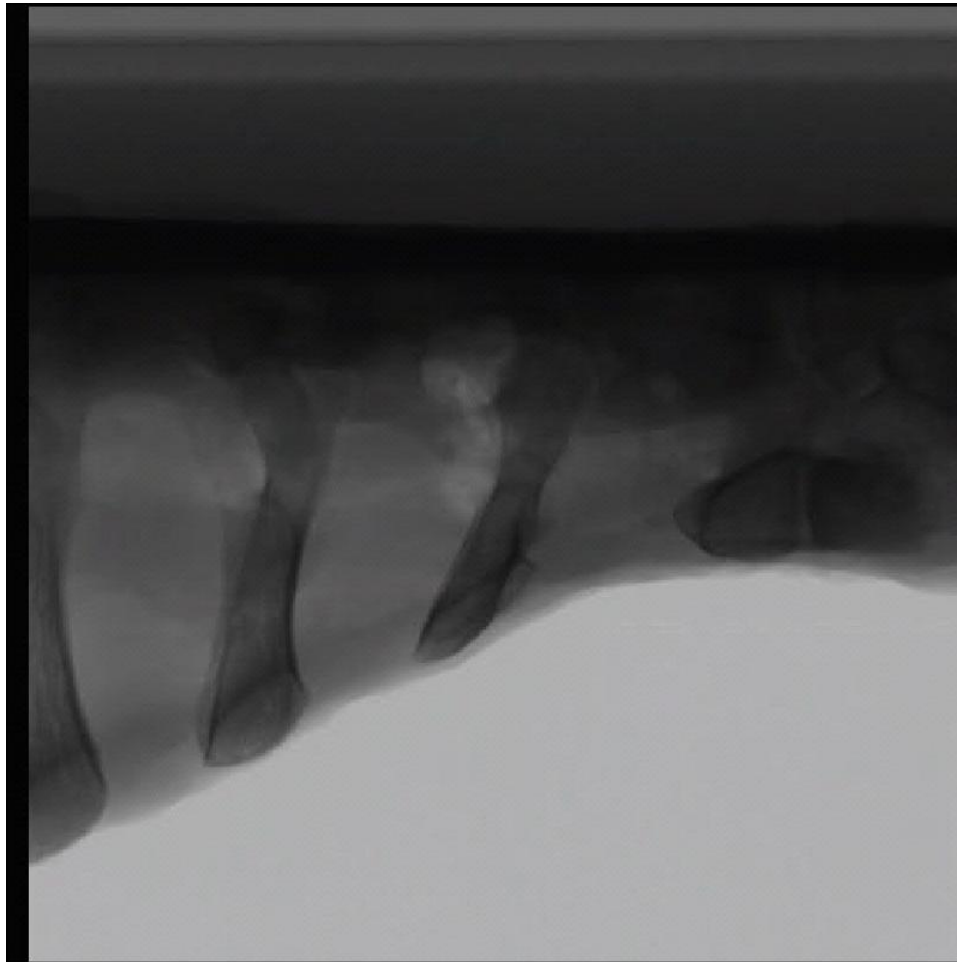
# Interventional 3D Angiography



# Principle of 3D Imaging from 2D X-ray



# Principle of 3D Imaging from 2D X-ray



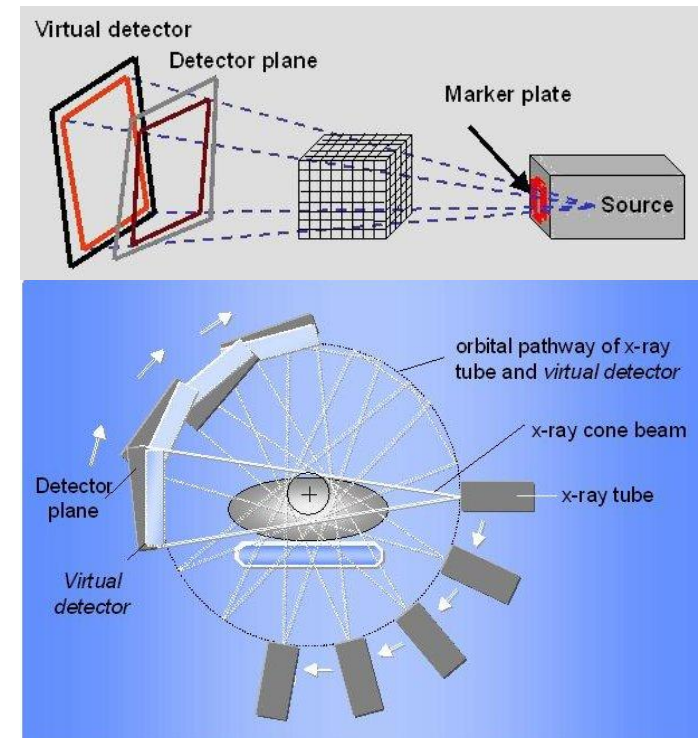
# Virtual Detector



**Rectifies X-ray images to simulate a pinhole camera with fixed intrinsic parameters:**

- Images are warped such that the markers are projected onto pre-defined positions
- planar transformation onto a real or virtual reference image
- Planar transformation is applied to the final projection matrix

$$P_i = H_{i \leftarrow ref} \cdot P_{ref} \cdot \begin{bmatrix} R_i & t_i \\ 0 & 1 \end{bmatrix}$$



# Non-reproducible C-arm Motion

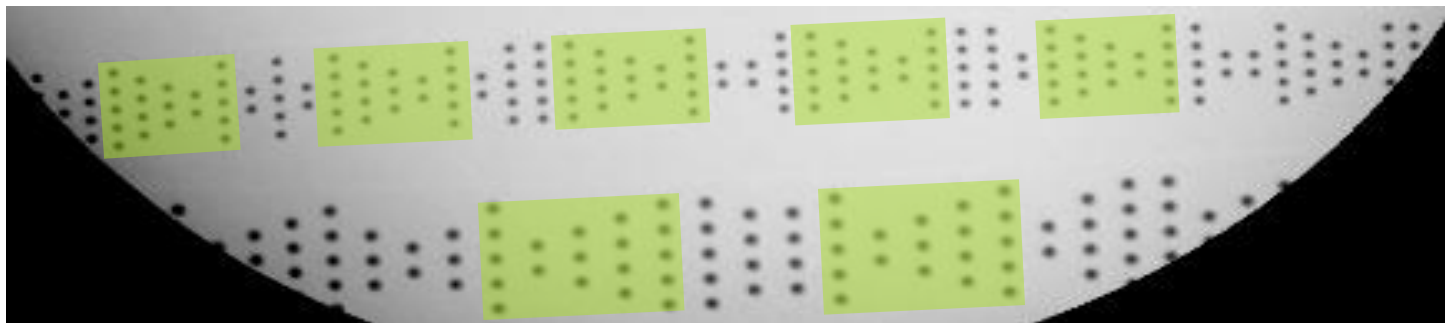
- Problems to deal with
  - Irregular and non-reproducible circular motion
  - Bending of C-frame (non-constant intrinsic parameters)
- Calibration procedure
  - Online
  - Use of X-ray calibration phantom not possible (smaller field of view, different anatomical regions)
  - Introduction of a secondary sensor system for estimation of projection geometry necessary
- Literature
  - Navab, Mitschke (MICCAI 99,00, CVPR 00, SPIE Med 00)

# Calibration Basics for X-ray

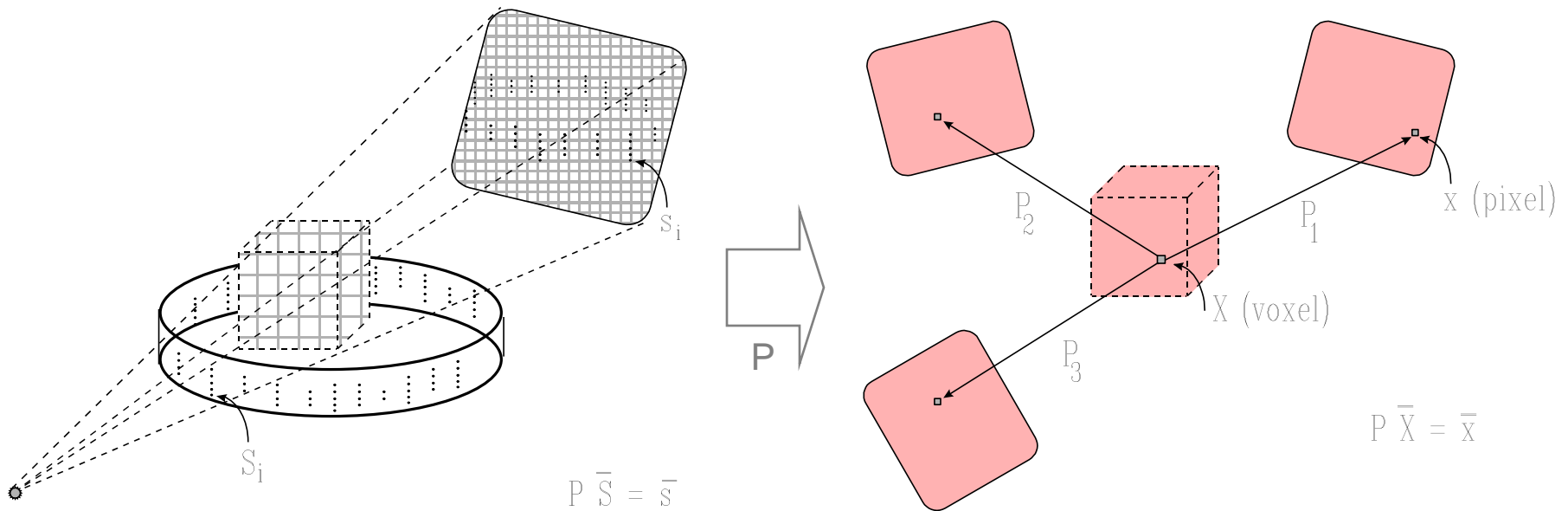
- Estimation of transformation “image” ← “world“
- Models
  - Physical parameters (distance focus – detector, distance to isocenter, pixel size, ...)
    - standard reconstruction algorithms (CT imaging) –
  - Projective parameters (estimation of projection matrix)
    - adapted reconstruction algorithms –
- Calibration procedure
  - Detection and identification of imaged marker structures (calibration phantom with known model) in the X-ray image

# Dynamic Calibration Phantom

- Steel bearings in cylindrical plastic frame
- Coded markers with „space” and „code” marker structures
- It is currently used for off-line calibration



# Reconstruction Using Projection Matrices



new 3D reconstruction algorithm based on  
approximate Feldkamp: ITMI, June 2000