

Computer Aided Medical Procedures II

Lecture Notes

Variational Deformable Registration

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1 Disclaimer

Dear Reader. These notes are a supplementary to the lecture *Computer Aided Medical Procedures II*, offered at the TU München by Prof. Dr. Nassir Navab. The notes are *neither* complete *nor* self contained and meaningful only combined with the visit of the lecture. Comments are gladly accepted at zikic@in.tum.de.

2 Outline

This short note briefly presents the basic steps involved in a standard method of performing deformable image registration by the variational approach.

What we present here is the probably the simplest version of a variational deformable registration algorithm, and well discussed in the literature. A good reference for further information are for example the paper by Weickert *et al.* [4] or the book on registration by Modersitzki [3]. For a general overview on registration of medical image, you might want to take a look at the book *Medical Image Registration* [1]

In this approach, all involved quantities (input images and the sought displacement field) are treated as continuous functions. The problem of deformable registration is modeled as minimizing an energy functional consisting of a dissimilarity between two images, subject to a smoothness condition imposed by a regularization term. Calculus of Variations is used in order to compute the so called Euler-Lagrange equations, corresponding to the minimization of the energy functional. The Euler-Lagrange equations present a necessary condition for the solution of the minimization problem. The resulting Euler-Lagrange partial differential equation (PDE) is non-linear. In order to solve this non-linear PDE, we replace it by a linear and time-dependent equation, by introducing an artificial time. The stationary state of the time dependent equation is considered as

solution of the original non-linear problem. Finally, the linear equation is discretized by Finite Difference method, such that the so obtained linear algebraic systems can be solved by standard solution methods.

3 Prerequisites

Image I is a function $I : \mathbb{R}^d \rightarrow \mathbb{R}$, which maps to every point of the image domain Ω an intensity value. The dimension is denoted by d , and is in our case $d = 2$ or $d = 3$.

Input images are the target image I_T and source image I_S . The source image I_S is to be deformed by a deformation function φ , such that it becomes similar to the target image I_T .

The Deformation Function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is defined by

$$\varphi = \text{Id} - u \quad (1)$$

or point-wise as

$$\varphi(\mathbf{x}) = \mathbf{x} - u(\mathbf{x}) \quad . \quad (2)$$

Here $u : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is the so called Displacement Function. Since we use the backward warping method, the displacement values are subtracted instead of being added. The displacement function is the unknown, which is to be computed by the deformation process.

Using the deformation function, we can define warped version of the source image is denoted by

$$I_S(\varphi) = I_S(\text{Id} - u) \quad , \quad (3)$$

or point-wise

$$I_S(\varphi(\mathbf{x})) = I_S(\mathbf{x} - u(\mathbf{x})) \quad . \quad (4)$$

This basically mean that the intensity of the image is not evaluated at the original point \mathbf{x} but at the new position $\mathbf{x} - u(\mathbf{x})$, according to the displacement function.

The single components of the vector-valued displacement field are denoted in the following way

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_d \end{bmatrix} \quad . \quad (5)$$

where the single u_i represent respective scalar-valued functions, that is $u_i : \mathbb{R}^d \rightarrow \mathbb{R}$.

4 The Model

We model the deformable registration problem as a minimization of a certain energy functional J , which measures the dissimilarity D between the target and

the warped image, while simultaneously imposing a regularization condition S onto the solution

$$J(I_T, I_S, u) = D(I_T, I_S, u) + \alpha S(u) . \quad (6)$$

Here, α is a scalar controlling the influence of the regularization term. In the following, we will drop the images I_T and I_S as arguments of the functionals.

As dissimilarity measure we choose the sum of squared differences

$$D(u) = \frac{1}{2} \int_{\Omega} (I_T(\mathbf{x}) - I_S(\mathbf{x} - u(\mathbf{x})))^2 d\mathbf{x} . \quad (7)$$

As regularization we use the so called diffusion regularization term

$$S(u) = \frac{1}{2} \sum_{i=1}^d \int_{\Omega} \|\nabla u_i(\mathbf{x})\|^2 d\mathbf{x} . \quad (8)$$

The model problem is solved by deriving the corresponding Euler-Lagrange equations. These are derived by the Calculus of Variations and present a necessary condition for the existence of an extremal point of the energy functional.

5 Euler-Lagrange Equations

Our model problem (6) can be written in a general form as

$$J(u) = \frac{1}{2} \int_{\Omega} F(\mathbf{x}_1, \dots, \mathbf{x}_d, u, u_{x_1}, \dots, u_{x_d}) d\mathbf{x} . \quad (9)$$

Please note that $u : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a vector valued function and that we denote the scalar-valued components of u by $u = [u_1 \dots u_d]^T$. On the other hand, by the notation u_{x_i} we understand the derivative of u with respect to the direction x_i , that is $u_{x_i} = du/dx_i$. The corresponding Euler-Lagrange equation has the form

$$0 = \frac{\partial F}{\partial u_j} - \sum_{i=1}^d \frac{d}{dx_i} \frac{\partial F}{\partial u_{j x_i}} \quad \forall j = 1 : d . \quad (10)$$

The derivation of the Euler-Lagrange equations can be found in standard literature, for example in Meyberg and Vachenaer [2].¹

Applying the general form of the Euler-Lagrange equation to our problem yields the equation

$$\mathbf{0} = \underbrace{(I_T(\mathbf{x}) - I_S(\mathbf{x} - u(\mathbf{x}))) \cdot \nabla I_S(\mathbf{x} - u(\mathbf{x}))}_{\text{results from } D} + \underbrace{(-\alpha \Delta u(\mathbf{x}))}_{\text{results from } S} . \quad (11)$$

¹This book however is in German. If anyone reading this has a good English reference, which contains a comprehensible derivation for $\mathbb{R}^m \rightarrow \mathbb{R}^n$ functions, I would really appreciate a short note.

The Laplacian of the vector field $u = [u_1 \dots u_d]$ is defined component-wise as $\Delta u = [\Delta u_1 \dots \Delta u_d]$. The definition of the Laplacian for the single, scalar-valued parts is $\Delta u_i = \sum_{j=1}^d \partial_{x_j x_j} u_i$.

With the following notation

$$f(u)(\mathbf{x}) = (I_T(\mathbf{x}) - I_S(\mathbf{x} - u(\mathbf{x}))) \cdot \nabla I_S(\mathbf{x} - u(\mathbf{x})) \quad (12)$$

$$A(u)(\mathbf{x}) = -\Delta u(\mathbf{x}) \quad , \quad (13)$$

we can write the resulting Euler-Lagrange equation as

$$0 = f(u) + \alpha A(u) \quad , \quad (14)$$

$$\alpha A(u) = -f(u) \quad . \quad (15)$$

For the derivation of the terms f and A , please refer also to the Appendix Section A.

The difficulty with the partial differential equation (PDE) (15) is that it is non-linear because of the dependency of the right-hand side force term f on the unknown displacement function u and on the input images I_T and I_S .

6 Quasilinearization and Time Discretization

One possible way of dealing with the non-linearity of the problem is to replace it by an iteration of linear problems. We briefly discuss two approaches, the Fix-Point Iteration, and the damped Fix-Point Iteration.

In the following, $u^{(i)}$ will denote the estimate of u in the i -th iteration.

6.1 Fix-Point Iteration

Starting from the initial estimate $u^{(0)} = 0$, the following equation is solved in order to get a new estimate $u^{(i+1)}$, (hopefully) until convergence is reached

$$\alpha A(u^{(i+1)}) = -f(u^{(i)}) \quad . \quad (16)$$

6.2 Relaxed Fix-Point Iteration

Starting from the initial estimate $u^{(0)} = 0$, the following equation is solved in order to get a new estimate $u^{(i+1)}$, (hopefully) until convergence is reached

$$u^{(i+1)} + \tau \alpha A(u^{(i+1)}) = u^{(i)} - \tau f(u^{(i)}) \quad . \quad (17)$$

The equation can also be written as

$$(\text{Id} + \tau \alpha A)(u^{(i+1)}) = u^{(i)} - \tau f(u^{(i)}) \quad , \quad (18)$$

with the identity operator Id .

7 Spatial Discretization by Finite Difference Method

The resulting linear PDE from Equation (18), which has to be solved in every iteration, has to be discretized in order to do this. To this end, we have to perform two steps:

1. Sampling of the involved functions, that is of the input images I_T , I_S and the displacement u . To this end, we employ a regular grid - in the simplest case such with a same spacing in all dimensions.
2. Discrete approximation of the continuous differential operators used in the PDE. For our model problem, only the first and second derivative along single dimensions are needed - by a combination of these, all operators can be assembled. The simplest possible approximation in our case is that by the Finite Difference method.

7.1 Approximations of the Differential Operators

For a scalar-valued function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, with the unit vectors in respective dimensions given by \mathbf{e}_i , and the uniform grid spacing h , the first and second derivative can be approximated in the following way by central difference approximation.

Using the following discrete derivatives, all operators used in the discussed algorithm can be implemented.

7.1.1 First Derivative

$$\frac{d}{dx_i} f(\mathbf{x}) = \frac{f(\mathbf{x} + h\mathbf{e}_i) - f(\mathbf{x} - h\mathbf{e}_i)}{2h}, \quad (19)$$

7.1.2 Second Derivative

$$\frac{d^2}{dx_i^2} f(\mathbf{x}) = \frac{f(\mathbf{x} + h\mathbf{e}_i) - 2f(\mathbf{x}) + f(\mathbf{x} - h\mathbf{e}_i)}{h^2}, \quad (20)$$

8 Solving the Linear Algebraic Systems

In the simple case of standard diffusion regularization, the resulting sparse linear systems can be solved by a variety of efficient methods. Among the discussed in literature are Conjugate Gradient, Multigrid, Fourier, and Additive Operator Splitting, compare for example [3].

For the exercise, we will simply use the backslash (`' \ '`) Matlab operator.

A Derivation of the Euler-Lagrange Equation

We derive here the terms f and A which constitute the Euler-Lagrange equation $0 = f(u) + \alpha A(u)$, compare Equation (15). The term f corresponds to the Sum of Squared Differences measure, and the term A results from the Diffusion regularization term. The derivation of the terms can be performed independently because of the linearity of the energy functional (6).

A.1 Sum of Squared Differences (SSD)

A.1.1 Energy Functional

The energy functional to be minimized is

$$D_{\text{SSD}}(u) = \int_{\Omega} \frac{1}{2} (I_T(\mathbf{x}) - I_S(\mathbf{x} - u(\mathbf{x})))^2 d\mathbf{x} . \quad (21)$$

It has the general form

$$D_{\text{SSD}} = \int_{\Omega} F_{\text{SSD}}(\mathbf{x}_1, \dots, \mathbf{x}_d, u) d\mathbf{x} , \quad (22)$$

with

$$F_{\text{SSD}} = \frac{1}{2} (I_T(\mathbf{x}) - I_S(\mathbf{x} - u(\mathbf{x})))^2 . \quad (23)$$

Note that F_{SSD} does not depend on derivatives of u .

A.1.2 Euler-Lagrange Derivation

Since the term F_{SSD} does not depend no derivatives of u , the general model from Equation (9) can be simplified, such that the corresponding part of the Euler-Lagrange equation simplifies from (10) to

$$f_{\text{SSD}}(u)(\mathbf{x}) = \frac{\partial F_{\text{SSD}}}{\partial u_j} \quad \forall j = 1 : d . \quad (24)$$

Evaluating this formula for yields for every dimension j we obtain the following equation by applying the chain rule (we use the definition $\varphi(\mathbf{x}) = \mathbf{x} - u(\mathbf{x})$)

$$f_{\text{SSD}}(u)(\mathbf{x}) = \frac{\partial}{\partial u_j} \left(\frac{1}{2} (I_T(\mathbf{x}) - I_S(\mathbf{x} - u(\mathbf{x})))^2 \right) \quad (25)$$

$$= \frac{1}{2} \cdot 2 \cdot (I_T(\mathbf{x}) - I_S(\varphi(\mathbf{x}))) \cdot \left(-\frac{\partial I_S}{\partial \varphi}(\varphi(\mathbf{x})) \right)^{\top} \frac{\partial \varphi}{\partial u_j}(\mathbf{x}) \quad (26)$$

$$= (I_T(\mathbf{x}) - I_S(\varphi(\mathbf{x}))) \cdot (-\nabla I_S)(\varphi(\mathbf{x}))^{\top} \left(\frac{\partial(I\mathbf{d} - u)}{\partial u_j}(\mathbf{x}) \right) \quad (27)$$

$$= (I_T(\mathbf{x}) - I_S(\varphi(\mathbf{x}))) \cdot (-\nabla I_S)(\varphi(\mathbf{x}))^{\top} (-\mathbf{e}_j) \quad (28)$$

$$= (I_T(\mathbf{x}) - I_S(\varphi(\mathbf{x}))) \cdot \frac{dI_S}{dx_j}(\varphi(\mathbf{x})) \quad (29)$$

$$= (I_T(\mathbf{x}) - I_S(\varphi(\mathbf{x}))) \cdot \frac{dI_S}{dx_j}(\mathbf{x} + u(\mathbf{x})) . \quad (30)$$

Here, \mathbf{e}_j denotes the unit vector in the j -th dimension, that is, the j -th entry is 1 and all others equal 0.

Taken together for all dimensions j , this give the following expression for every point of the image domain

$$f_{\text{SSD}}(u)(\mathbf{x}) = (I_T(\mathbf{x}) - I_S(\mathbf{x} - u(\mathbf{x}))) \cdot \nabla I_S(\mathbf{x} - u(\mathbf{x})) \quad . \quad (31)$$

A.2 Diffusion Regularization

A.2.1 Energy Functional

The energy functional is defines such that large gradients of every scalar-valued component u_i of u are penalized

$$S_{\text{DIFF}}(u) = \int_{\Omega} \frac{1}{2} \sum_{k=1}^d \|\nabla u_k(\mathbf{x})\|^2 \, d\mathbf{x} \quad . \quad (32)$$

The functional has the general form

$$S_{\text{DIFF}} = \int_{\Omega} F_{\text{DIFF}}(\mathbf{x}_1, \dots, \mathbf{x}_d, u_{x_1}, \dots, u_{x_d}) \, d\mathbf{x} \quad , \quad (33)$$

with

$$F_{\text{DIFF}} = \frac{1}{2} \sum_{k=1}^d \|\nabla u_k(\mathbf{x})\|^2 \quad (34)$$

$$= \frac{1}{2} \sum_{k=1}^d \langle \nabla u_k(\mathbf{x}), \nabla u_k(\mathbf{x}) \rangle \quad (35)$$

$$= \frac{1}{2} \sum_{k=1}^d \sum_{l=1}^d u_{kx_l}(\mathbf{x})^2 \quad . \quad (36)$$

Please note that F_{DIFF} does not depend directly on the displacement function u itself, but only on the derivatives u_{jx_i} in the single directions x_i .

A.2.2 Euler-Lagrange Derivation

Since the term F_{DIFF} does not depend on the function u itself, the general model from Equation (9) can be simplified, such that the corresponding Euler-Lagrange equation simplifies from (10) to

$$A_{\text{DIFF}}(u)(\mathbf{x}) = - \sum_{i=1}^d \frac{d}{dx_i} \frac{\partial F}{\partial u_{jx_i}} \quad \forall j = 1 : d \quad . \quad (37)$$

Evaluating this formula for yields for every dimension j of the we obtain the following equation

$$A_{\text{DIFF}}(u)(\mathbf{x}) = - \sum_{i=1}^d \frac{d}{dx_i} \frac{\partial F}{\partial u_{j x_i}} \quad (38)$$

$$= - \sum_{i=1}^d \frac{d}{dx_i} \frac{\partial}{\partial u_{j x_i}} \frac{1}{2} \sum_{k=1}^d \sum_{l=1}^d u_{k x_l}(\mathbf{x})^2 \quad (39)$$

$$= - \frac{1}{2} \sum_{i=1}^d \frac{d}{dx_i} \frac{\partial}{\partial u_{j x_i}} \sum_{k=1}^d \sum_{l=1}^d u_{k x_l}(\mathbf{x})^2 \quad (40)$$

$$= - \frac{1}{2} \sum_{i=1}^d \frac{d}{dx_i} \left(2 \cdot u_{j x_i}(\mathbf{x}) \right) \quad (41)$$

$$= - \sum_{i=1}^d \frac{d}{dx_i} u_{j x_i}(\mathbf{x}) \quad (42)$$

$$= - \sum_{i=1}^d \frac{d^2}{dx_i^2} u_j(\mathbf{x}) \quad (43)$$

$$= -\Delta u_j(\mathbf{x}) \quad (44)$$

Here, Δ denotes the Laplacian operator.

By using the fact that the Laplacian for vector-valued functions is defined as the component-wise Laplacian of the single scalar-valued components (that is $\Delta u = [\Delta u_1 \dots \Delta u_d]$) we can give the following expression for every point of the image domain

$$A_{\text{DIFF}}(u)(\mathbf{x}) = -\Delta u(\mathbf{x}) \quad , \quad (45)$$

which represents the part of the Euler-Lagrange equation corresponding to the diffusion regularization term.

References

- [1] Joseph V. Hajnal, Derek L.G. Hill, and David J. Hawkes, editors. *Medical Image Registration*. CRC Press, 2001.
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- [4] J. Weickert and C. Schnörr. A theoretical framework for convex regularizers in pde-based computation of image motion. *International Journal of Computer Vision*, 45(3):245–264, 2001.