

2 Projective Geometry

2.1 Overview

In this chapter we will introduce the so called *Projective Geometry*, a way of describing the 2- and 3-dimensional space and the entities therein, which will be used throughout the lecture. Therefore it is essential to get a good understanding of this fundamental concept.

In the first Section we will give a short introduction to the 2D projective space. Here we explain what the 2D projective space is and how the basic entities therein (points, lines) can be described using the appropriate notation (homogeneous coordinates). We also introduce the relationship between these entities.

This introduction takes an approach of geometrically motivated explanation and lacks strict formal derivation and completeness. We hope that the approach is intuitive and gives an understanding of the subject that enables the reader to accept the mathematical derivation.

2.2 2D Projective Space \mathbb{P}^2

In the course of the lecture we will use the so called homogeneous coordinates for describing 2D and 3D. In inhomogeneous coordinates a point p in 2D is represented as a coordinate pair $p = (x, y)$. This is a “common” representation of a plane as a subset of \mathbb{R}^2 . Now we introduce the homogeneous coordinates in which a point p is represented as $p = (x, y, z)$. If we set $z = 1$ then the coordinates x and y are the same as in \mathbb{R}^2 .

In this representation we set that two points are the same / equivalent if they only differ in scale. Hence $p = (x, y, z) = k(x, y, z)$.

This way every vector represents an equivalence class of vectors. The vectors of this equivalence class only differ in scale. This class is described as a homogeneous vector.

The set of all homogeneous vectors from \mathbb{R}^3 (except $(0, 0, 0)$) form the 2D projective space \mathbb{P}^2 .

A useful model for thinking about \mathbb{P}^2 is imagining a 2D homogeneous point as a point P that results from an intersection of a line p and a plane π parallel to the xy -plane (Figure 2.1). If π goes through $z = 1$ then the x and y coordinates of p are the same as in the \mathbb{R}^2 plane. The difference is that every intersection of p with any plane parallel to the xy -plane represents the same point. Hence the line p represents the class of all representations of the point P and can be interpreted as the homogeneous vector P .

In projective space, lines as well as points are represented as homogeneous vectors. Using the upper model, lines are intersections of planes parallel to the xy -plane and an arbitrary plane (Figure 2.3).

So lines as well as points are represented in the homogeneous 2D projective space \mathbb{P}^2 as single vectors. This may seem weird at the beginning but it has many advantages and

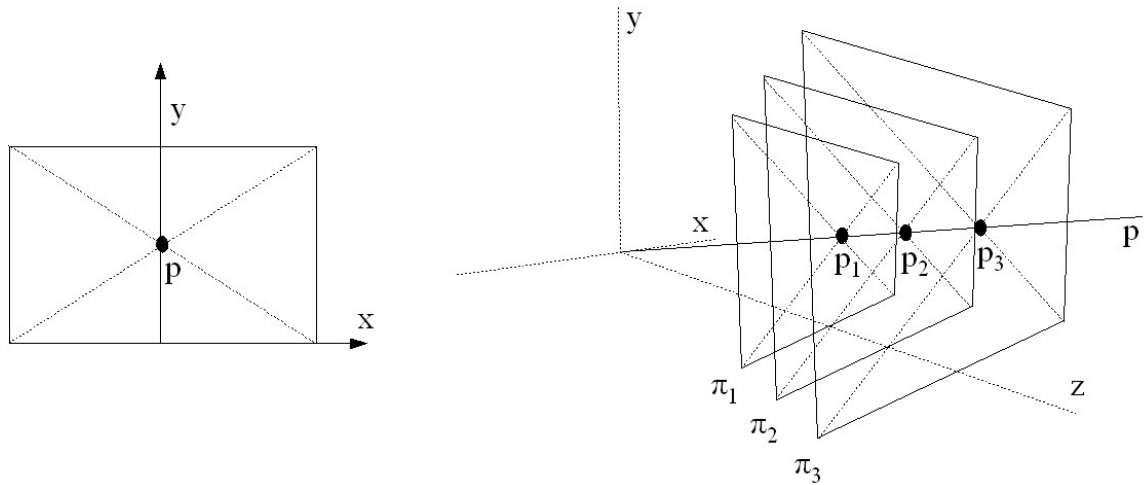


Figure 2.1: Since all points in \mathbb{P}^2 are up to scale, one point represents an equivalence class of points

there is a nice geometrical explanation for all of this using the model introduced above. Now here it comes ...

2.2.1 Representation of Basic Entities

2.2.1.1 Representation of Points

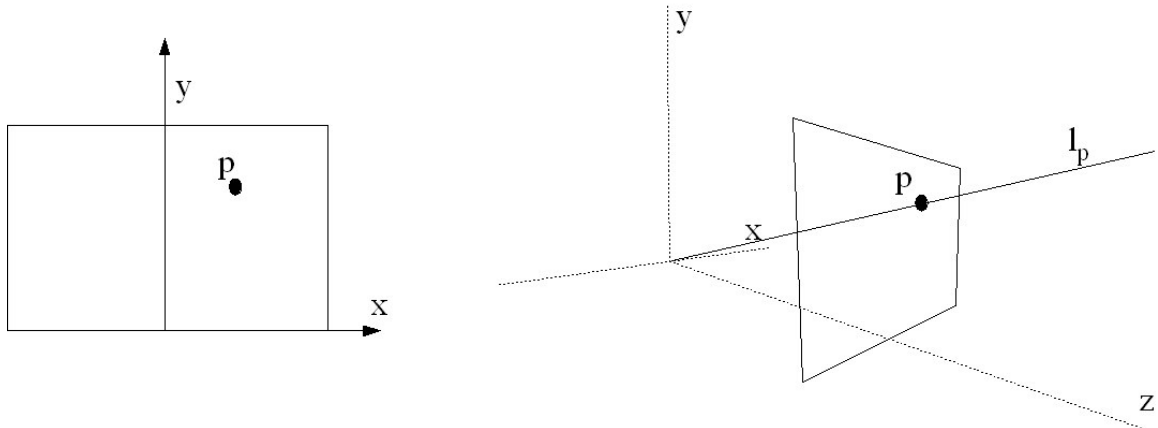
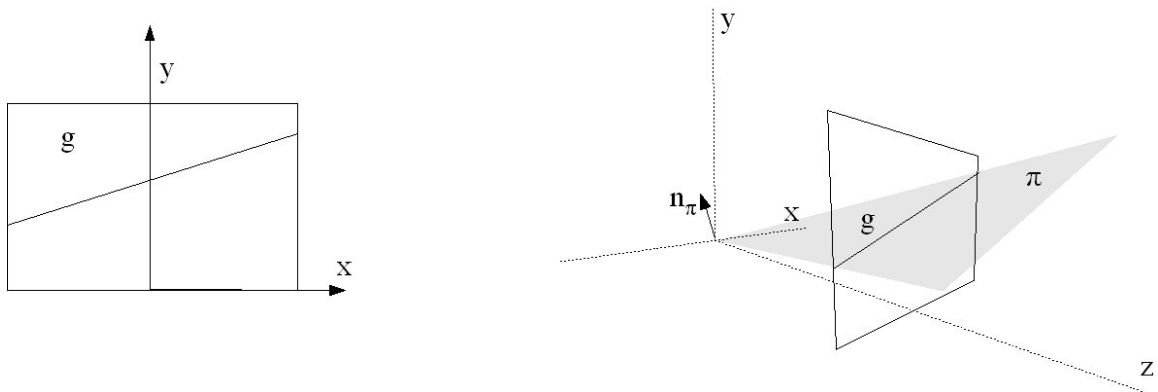
This is easy. As we saw above, a homogeneous point p can be represented by a line in \mathbb{R}^3 intersecting a plane parallel to the xy -plane. This line always goes through the origin. So we can describe it by providing its direction. And a direction is simply described by a \mathbb{R}^3 -vector. So \mathbb{R}^3 -vectors represent homogeneous points. This is illustrated once again in Figure 2.2.

2.2.1.2 Representation of Lines

Now we come to the more interesting case. Lines in homogeneous coordinates can be described by vectors from \mathbb{R}^3 as well. Strange, isn't it? But fortunately there is a nice explanation for this as well.

A homogeneous line g can be described by a plane π in \mathbb{R}^3 intersecting a plane parallel to the xy -plane (Figure 2.3). Please notice that π goes through the origin. So the distance of π to the origin is 0. This way a vector n_π normal to π is enough to describe the plane π . So this vector can be interpreted as the homogeneous line g .

This observation also enables us to easily compute the direction of a line in \mathbb{P}^2 . A direction of a line g d_g is normal to the vector n_π . Furthermore the direction is coplanar to the xy -plane since g lies in a plane parallel to the xy -plane. Hence d_g is normal to the projection of n_π onto the xy -plane. And the projection of $n_\pi = (n_{\pi x}, n_{\pi y}, n_{\pi z})$ onto the xy -plane is $(n_{\pi x}, n_{\pi y}, 0)$. All this together yields that the direction of the line g in \mathbb{P}^2 is

Figure 2.2: Representation of a point p in \mathbb{R}^2 and in \mathbb{P}^2 Figure 2.3: Representation of a line g in \mathbb{R}^2 and in \mathbb{P}^2

$d_g = (n_{\pi y}, -n_{\pi x}, 0)$. In inhomogeneous coordinates / in \mathbb{R}^2 the direction of g would be $(n_{\pi y}, -n_{\pi x})$. From this argument we can also intuitively see that all directions in \mathbb{P}^2 have $z = 0$ since all the lines are coplanar to the xy -plane.

2.2.2 Relationships between Entities

2.2.2.1 Intersection of Lines

The nice thing about the way of thinking about points and lines described above is that it also makes those formulas for the intersection of lines and a line through two points geometrically clearer.

Suppose we are given two lines g_1 and g_2 . Then from the theory we now that the cross-product of the two lines $p = g_1 \times g_2$ gives the point p at which the lines intersect. This doesn't always seem clear at the first glance.

Let's take a look at the Figure 2.4. We have the planes π_1 and π_2 which "generate" the

lines g_1 and g_2 and at the \mathbb{R}^3 -vectors n_1 and n_2 normal to the planes. As described above, these vectors are exactly the vectors which are describing the lines. So the cross-product of g_1 and g_2 is actually the cross-product of n_1 and n_2 ($v_p = g_1 \times g_2 = n_1 \times n_2$). Since the result v_p of the cross-product of two vectors n_1 and n_2 is a vector orthogonal to the two vectors n_1 and n_2 we get a v_p which describes a line at which the planes π_1 and π_2 are intersecting. And this is exactly the line which “generates” the point p .

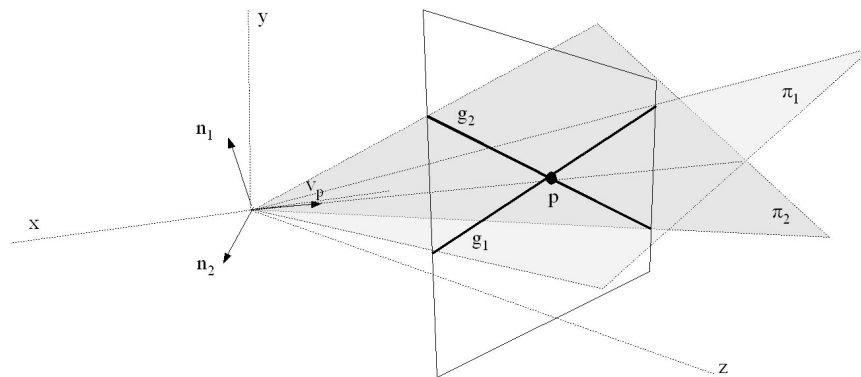


Figure 2.4: The intersection point p of lines g_1 and g_2 can be presented as their cross-product $p = g_1 \times g_2$.

2.2.2.2 Line through Two Points

Another formula of interest is that for a line g going through two points p_1 and p_2 . Here again we have $g = p_1 \times p_2$. How comes?

Please do take a look at the Figure 2.5. We have given the two points p_1 and p_2 . We also have the corresponding \mathbb{R}^3 -vectors / lines v_1 and v_2 which “generate” the points. Please do notice that two lines define one plane. Let’s call this plane π . This plane is obviously the plane which generates the line g . Anyway, the cross-product of v_1 and v_2 ($n_\pi = v_1 \times v_2$) gives a vector normal to the two vectors v_1 and v_2 and thus normal to the plane π . Since the plane normal represents the line “generated” by the plane ($n_\pi = g$) we have hereby shown that $g = p_1 \times p_2$.

Neat, isn’t it?

2.2.2.3 Incidence of Points and Lines

Given the point p and a line g . When does the line goes through the point, or the way round, when does the point lie on the line? In other words, when are the point and the line incident to each other?

Once again, when we take a look at the vectors n_π and v_p which “generate” the line and the point (Figure 2.6). It’s easy to see that those two lines have to be orthogonal to each other if the point and the line are incident. So we get the result that a line g and the point p are incident if and only if $p^T g = 0$ holds.

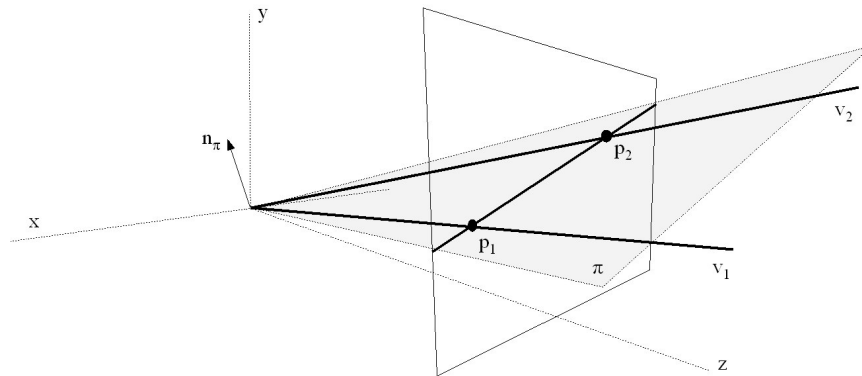


Figure 2.5: The line g going through points p_1 and p_2 can be represented as cross-product of the point vectors ($g = p_1 \times p_2$).

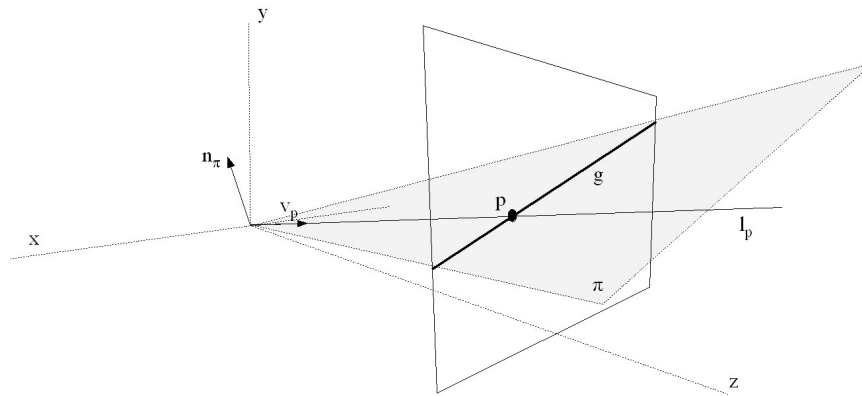


Figure 2.6: The line g is incident to the point p if and only if $p^T g = 0$.

2.2.2.4 Line at Infinity

From the theory we know that parallel lines intersect at a point at infinity also called an *ideal point*. In \mathbb{P}^2 all ideal points lie at one line, called the line at infinity. Algebraically, it's easy to derive that ideal points got to have the last coordinate equal to zero so generally a point $x = (x, y, 0)$ is a point at infinity. Then it's easy to see that the line incident to any ideal point must be $l_{\text{inf}} = (0, 0, 1)$, since $x^T l = 0$.

Here we want to derive a geometrically more intuitive approach. Suppose we have two parallel lines g_1 and g_2 given (Figure 2.7). In \mathbb{P}^2 those g_1 and g_2 will intersect at a point at infinity. The same holds for any other two parallel lines like h_1 and h_2 . So all parallel lines intersect in ideal points. We want to show that all these ideal points lie at the line at infinity $(0, 0, 1)$.

In our model however, where we use \mathbb{R}^3 to represent \mathbb{P}^2 those parallel lines do not intersect. Please recall that a line in \mathbb{P}^2 can be imagined as generated by an intersection of a plane π with a plane parallel to the xy -plane. Since we are looking for the line at infinity

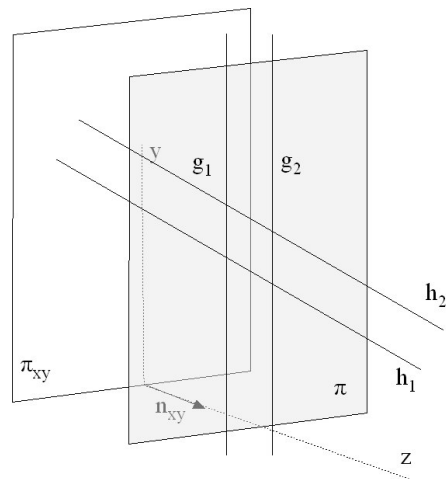


Figure 2.7: The line at infinity at which all parallel lines intersect is $(0, 0, 1)$.

we are looking for a line that is generated by a plane π that intersects a plane parallel to the xy -plane at infinity. This is clearly the case when π is parallel to the xy -plane. So we get that the line at infinity l_{inf} is “generated” by a plane π parallel to the xy -plane. And because such a plane π is defined through its normal vector n_{xy} which is $n_{xy} = (0, 0, 1)$ we get the result that the line at infinity is $l_{\text{inf}} = (0, 0, 1)$.