

## 9.5 Conics and Quadrics

In the lecture we will use *conics* and *quadric* as geometrical entities. These concepts are usually not so well known and also not always very intuitive. We experienced that this is especially confusing at the beginning because the the introduction to projective space is done at the same time. So it is sometimes better for the beginning, not to try to interpret these entities but only to know (and accept) the mathematical definitions and how they are dealt with.

The literature use in this section is Hartley and Zisserman [3] (p. 30, p. 73) and Meyberg and Vachenaer [4] (Chapter 6, Section 7).

In order to define conics and quadrics we will first introduce the so called *quadratic forms*. The conic and quadric are the special cases of quadratic forms for the three and four dimensional case respectively.

**Definition 9.27 (Quadratic Form)** *We can express any quadratic function  $f$  by a square matrix  $A$*

$$f(x) = x^T A x$$

*It can be shown that we can assume that  $A$  is symmetric without loss of generality.*

*Given a symmetric matrix  $A$  then we denote the expression  $f(x) = x^T A x$  as Quadratic Form*

The fact that the matrix  $A$  in the quadratic form  $f(x) = x^T A x$  is symmetric has some nice properties. Given a symmetric matrix  $A \in \mathbb{R}^{n \times n}$ , we have:

- All eigenvalues  $\lambda_i$  of  $A$  are real numbers ( $\lambda_i \in \mathbb{R}$ ).
- Eigenvectors belonging to different eigenvalues of  $A$  are orthogonal.
- algebraic and geometric multiplicities are the same.

This makes it possible for us to transform any conic or quadric to a diagonal matrix. This is necessary for the classification of the conics and quadrics later on.

### 9.5.1 Conics

**Definition 9.28 (Conic)** *A Conic is a curve described by a quadratic equation in a plane. For homogeneous coordinates this is a set of all points  $x \in \mathbb{R}^3$  which satisfy the equation*

$$0 = x^T C x$$

*where  $C$  is a symmetric  $3 \times 3$  matrix.*

*The matrix  $C$  is enough to describe a certain conic. Explicitly, it is*

$$\text{For } x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } C = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

*Using this we can write the equation for the conic also as*

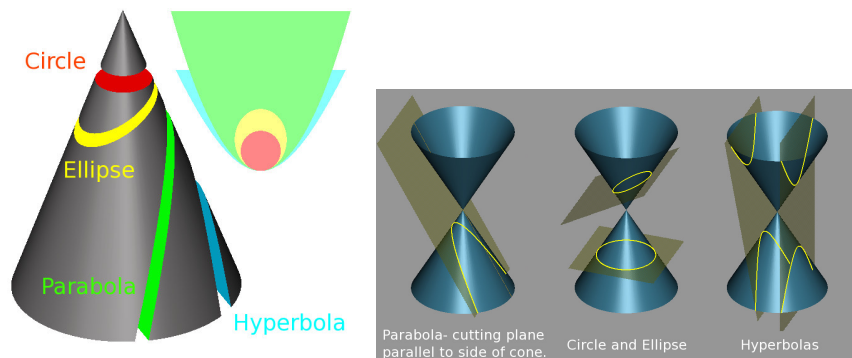
$$x^T C x = ax^2 + bxy + cy^2 + dxz + exz + fz^2$$

We can see that the conic is a special case of the quadratic form. Thus all properties of quadratic forms hold for conics as well.

The conic defining matrix  $C$  has obviously 6 independent parameters. Since we are using the projective geometry framework however, all entities are up to scale. Hence a conic has only 5 degrees of freedom. *That is, 5 points define a conic.*

### 9.5.1.1 Intuition and Classification - Euclidean vs. Projective Framework

The geometric meaning of the conic is following. A *conic* is a quadratic curve in a plain that we get by intersecting a cone and a plane. Depending on the way we intersect the plane with the cone we get four different types of the conic. The curves we get can be *open* or *closed*.



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Figure 9.1: Conic classification in the Euclidean Framework. Images: <http://en.wikipedia.org/wiki/Conic>

If the plane is parallel to the axis of the cone, we get a *circle*, which is a closed curve. If the plane is not parallel any more but the curve we get by the intersection is still closed, then the curve is an *ellipse*. When the plane is parallel to the line which defines the cone (i.e. to the side of the cone) we get an open curve named *parabola*. And when the angle gets even bigger and the plane is not parallel to the side of the cone any more and the curve is open, it is named *hyperbola*.

These are the so called nondegenerate conics. Then there are also the so called *degenerate conics* which are intersections of the cone and the plane going through the tip of the cone - resulting in either a point or two lines.

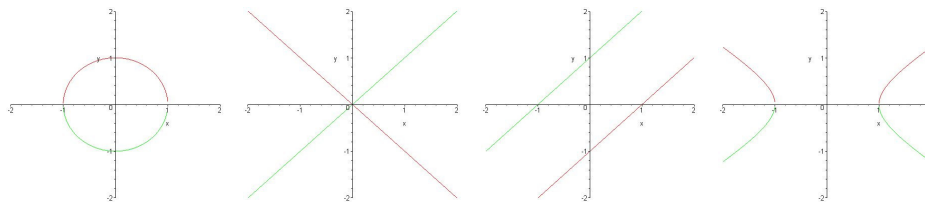


Figure 9.2: Four conic examples

This classical classification however is only valid for the *Euclidean geometry framework*. It

can be shown however, that the all nondegenerate four conics types are equivalent under projective transformations.

It is extremely important to always keep in mind which framework, the Euclidean or the projective you are using. Especially at the beginning of the Computer Vision lecture, this is something that is really difficult to get used to.

Using the diagonalization property of the quadric forms we can classify the conics for the *projective framework* by the form of the diagonals corresponding to them.

Diagonal of $C$	Conic Type
(1,1,1)	This is an improper conic. It has no real points solving the equation.
(1,1,-1)	This is a circle. In projective geometry it is equivalent to the circle, ellipse, parabola and hyperbola form the Euclidean geometry framework.
(1,1,0)	This conic is one single point.
(1,-1,0)	This gives us the conic consisting of two lines.
(1,0,0)	One single line that is counted twice.

### 9.5.2 Quadrics

**Definition 9.29 (Quadric)** A Quadric is a set of all points  $x \in \mathbb{R}^4$  which satisfy the equation:

$$0 = x^T Q x$$

where  $Q$  is a symmetric  $4 \times 4$  matrix. So again, the quadric is a special case of the quadratic forms.

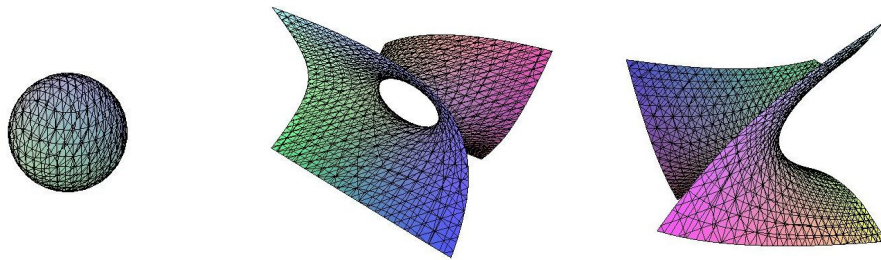


Figure 9.3: Three quadric examples