

1 The Jacobian for Minimization of Geometric Reprojection Error

1.1 Introduction

For estimation of a 2D homography H that takes a set of points $\{\vec{x}_i\}$ to their corresponding points $\{\vec{x}'_i\}$, i.e. $\vec{x}'_i = H\vec{x}_i$, different methods are proposed. Each method tries to minimize an error that occurs when the point sets and their correspondences are determined (measurement errors). Linear solutions like the Direct Linear Transformation Algorithm have the disadvantage that they minimize just an algebraic error function and do not consider geometric error minimization. For geometric error minimization of the homography (non-linear solutions), three cases can be distinguished. First, one constraints the problem to one point set, i.e. an error just arises in $\{\vec{x}'_i\}$, which is minimized taking the entries of H as parameters for minimization iteration. Second, the geometric error can occur in both point sets, but the minimization parameter also just consists of the entries of H . This is called the *symmetric transfer error*.

Third, one assumes the error to occur in both point sets and extends the parameter vector for minimization iteration with one point set such that the measurements can also be altered in order to get an optimal solution. This is called the *geometric reprojection error*, which, in mathematical terms, minimizes the error (cost) function

$$\sum_i (d(\vec{x}_i, H\vec{x}_i) + d(\vec{x}'_i, H^{-1}\vec{x}'_i))$$

For minimizing this error, an iteration algorithm has to be chosen such as Newton or Levenberg-Marquardt. Both algorithms take a parameter vector P and error function ϵ and minimize the error that occurs for $f(P)$, where f is a function that calculates the homography (in this case) for the point correspondences. In fact, both methods use the first derivative of that function $\frac{\partial f}{\partial P}$.

This derivative is called the *Jacobian*. In the following, the Jacobian for the above mentioned problem will be developed.

1.2 Notation

The homography $H \in \mathbb{R}^{3 \times 3}$ can be written as a 9-vector stacking all the entries of H in one column:

$$\vec{h} = \begin{pmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{pmatrix}$$

Moreover, it will also be written in a 3-vector where each entry represents a row in H :

$$H = \begin{pmatrix} \vec{h}^{1T} \\ \vec{h}^{2T} \\ \vec{h}^{3T} \end{pmatrix}$$

Points \vec{x} are usually 3-vectors representing homogeneous coordinates:

$$\vec{x} = \begin{pmatrix} x \\ y \\ w \end{pmatrix}$$

Points with a tilde will represent their respective inhomogeneous pendants:

$$\vec{\tilde{x}} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \frac{x}{w} \\ \frac{y}{w} \end{pmatrix}$$

1.3 Mathematical Problem Statement

Given is a set of point correspondences $\{\vec{x}_i \leftrightarrow \vec{x}'_i\}$ and a homography H . Determine the Jacobian of the function

$$f : (\vec{h}, \vec{x}_1, \dots, \vec{x}_n) \rightarrow (\vec{x}'_1, \vec{x}'_1, \dots, \vec{x}'_n, \vec{x}'_n)$$

where $\vec{x}'_i = H\vec{x}_i$.

The Jacobian will be built as defined for partial derivatives. For a function

$$f = \begin{pmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_m(x_1, \dots, x_n) \end{pmatrix}$$

the Jacobian is defined as

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

1.3.1 Derivative with respect to \vec{x}

For one point correspondence $\vec{x} \leftrightarrow \vec{x}'$ one can build the Jacobian with respect to \vec{x} :

$$\begin{aligned} \frac{\partial \vec{x}'}{\partial \vec{x}} &= \frac{\partial(\vec{x}', \vec{y}')}{\partial(x, y, w)} = \frac{\partial \left(\frac{\vec{h}^{1T} \vec{x}}{\vec{h}^{3T} \vec{x}}, \frac{\vec{h}^{2T} \vec{x}}{\vec{h}^{3T} \vec{x}} \right)}{\partial \vec{x}} = \begin{pmatrix} \frac{\vec{h}^{1T} \vec{h}^{3T} \vec{x} - \vec{h}^{3T} \vec{h}^{1T} \vec{x}}{(\vec{h}^{3T} \vec{x})^2} \\ \frac{\vec{h}^{2T} \vec{h}^{3T} \vec{x} - \vec{h}^{3T} \vec{h}^{2T} \vec{x}}{(\vec{h}^{3T} \vec{x})^2} \end{pmatrix} \\ &= \frac{1}{\vec{h}^{3T} \vec{x}} \begin{pmatrix} \frac{\vec{h}^{1T} \vec{h}^{3T} \vec{x}}{\vec{h}^{3T} \vec{x}} - \frac{\vec{h}^{3T} \vec{h}^{1T} \vec{x}}{\vec{h}^{3T} \vec{x}} \\ \frac{\vec{h}^{2T} \vec{h}^{3T} \vec{x}}{\vec{h}^{3T} \vec{x}} - \frac{\vec{h}^{3T} \vec{h}^{2T} \vec{x}}{\vec{h}^{3T} \vec{x}} \end{pmatrix} = \frac{1}{w'} \begin{pmatrix} \vec{h}^{1T} - \vec{x}' \vec{h}^{3T} \\ \vec{h}^{2T} - \vec{y}' \vec{h}^{3T} \end{pmatrix} \end{aligned}$$

Since $\vec{h}^{iT} \vec{x}$ are scalars they can be canceled out in the fraction. $\frac{\partial \vec{x}'}{\partial \vec{x}}$ will further be called $J_{\vec{x}}$ ($\in \mathbb{R}^{2 \times 3}$).

1.3.2 Derivative with respect to \vec{h}

For one point correspondence $\vec{x} \leftrightarrow \vec{x}'$ one can build the Jacobian with respect to \vec{h} :

$$\begin{aligned} \frac{\partial \vec{x}'}{\partial \vec{h}} &= \frac{\partial(\vec{x}', \vec{y}')}{\partial(\vec{h}^{1T}, \vec{h}^{2T}, \vec{h}^{3T})} = \frac{\partial \left(\frac{\vec{h}^{1T} \vec{x}}{\vec{h}^{3T} \vec{x}}, \frac{\vec{h}^{2T} \vec{x}}{\vec{h}^{3T} \vec{x}} \right)}{\partial \vec{h}} = \begin{pmatrix} \frac{\vec{x}^T}{\vec{h}^{3T} \vec{x}} & 0 & \frac{-\vec{x}^T \vec{h}^{1T} \vec{x}}{(\vec{h}^{3T} \vec{x})^2} \\ 0 & \frac{\vec{x}^T}{\vec{h}^{3T} \vec{x}} & \frac{-\vec{x}^T \vec{h}^{2T} \vec{x}}{(\vec{h}^{3T} \vec{x})^2} \end{pmatrix} \\ &= \frac{1}{w'} \begin{pmatrix} \vec{x}^T & 0 & -\vec{x}' \vec{x}^T \\ 0 & \vec{x}^T & -\vec{y}' \vec{x}^T \end{pmatrix} \end{aligned}$$

This derivative will be further called $J_{\vec{h}}$ ($\in \mathbb{R}^{2 \times 9}$).

1.3.3 Jacobian of f

Now, $\frac{\partial f}{\partial P}$ can be determined. Here, we have $9 + 2n$ variables in P and $2n$ functions f_i :

$$f(P) = f(\vec{h}, \vec{x}_1, \dots, \vec{x}_n) = \begin{pmatrix} \tilde{x}_1 \\ \tilde{y}_1 \\ \tilde{x}'_1 \\ \tilde{y}'_1 \\ \vdots \\ \tilde{x}_n \\ \tilde{y}_n \\ \tilde{x}'_n \\ \tilde{y}'_n \end{pmatrix} = \begin{pmatrix} \vec{z}_1 \\ \vec{z}'_1 \\ \vdots \\ \vec{z}_n \\ \vec{z}'_n \end{pmatrix} \begin{matrix} \leftarrow f_1 \\ \leftarrow f_2 \\ \vdots \\ \leftarrow f_{2n-1} \\ \leftarrow f_{2n} \end{matrix}$$

Here, we can use the inhomogeneous coordinates of \vec{x} if we assure that \vec{x}_i is in the $z = 1$ -plane, i.e. $w_i = 1$ of vector $\vec{x}_i = (x_i, y_i, w_i)^T$. However, the derivatives of the entries have to be built via homogeneous coordinates of \vec{x}_i to assure that also the points \vec{x}'_i stay in the $z = 1$ -plane (by division by w'_i) as showed above.

$$J = \frac{\partial f}{\partial P} = \begin{pmatrix} \frac{\partial \tilde{x}_1}{\partial h} & \frac{\partial \tilde{x}_1}{\partial \tilde{x}_1} & \dots & \frac{\partial \tilde{x}_1}{\partial \tilde{x}_n} \\ \frac{\partial \tilde{x}'_1}{\partial h} & \frac{\partial \tilde{x}'_1}{\partial \tilde{x}_1} & \dots & \frac{\partial \tilde{x}'_1}{\partial \tilde{x}_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \tilde{x}_n}{\partial h} & \frac{\partial \tilde{x}_n}{\partial \tilde{x}_1} & \dots & \frac{\partial \tilde{x}_n}{\partial \tilde{x}_n} \\ \frac{\partial \tilde{x}'_n}{\partial h} & \frac{\partial \tilde{x}'_n}{\partial \tilde{x}_1} & \dots & \frac{\partial \tilde{x}'_n}{\partial \tilde{x}_n} \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{2 \times 9} & \mathbf{1}_{w_1} & \dots & \mathbf{0}_{2 \times 3} \\ J_{\vec{h}}(\vec{z}'_1) & J_{\vec{x}_1}(\vec{z}'_1) & \mathbf{0}_{2 \times 3} & \dots & \mathbf{0}_{2 \times 3} \\ \mathbf{0}_{2 \times 9} & \mathbf{0}_{2 \times 3} & \mathbf{1}_{w_2} & \mathbf{0}_{2 \times 3} & \dots & \mathbf{0}_{2 \times 3} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0}_{2 \times 9} & \mathbf{0}_{2 \times 3} & \dots & \mathbf{0}_{2 \times 3} & \mathbf{1}_{w_n} & \mathbf{0}_{2 \times 3} \\ J_{\vec{h}}(\vec{z}'_n) & \mathbf{0}_{2 \times 3} & \dots & \mathbf{0}_{2 \times 3} & J_{\vec{x}_n}(\vec{z}'_n) & \mathbf{0}_{2 \times 3} \end{pmatrix}$$

J has dimension $4n \times (9 + 6n)$ and $\mathbf{1}_{w_i}$ is $\frac{1}{w_i} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

References: Hartley / Zisserman pp. 145 - 147, 607