



Visualize Pose Error Uncertainty

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28. November 2005

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Variance and Std Deviation

- Variance:

$$\begin{aligned} \text{Var}(x) &= \sum_{i=1}^n P(x_i)(x_i - \bar{x})^2 \\ &= E[(x_i - \bar{x})^2] \\ &= E[x^2] - E[x]^2 \end{aligned}$$

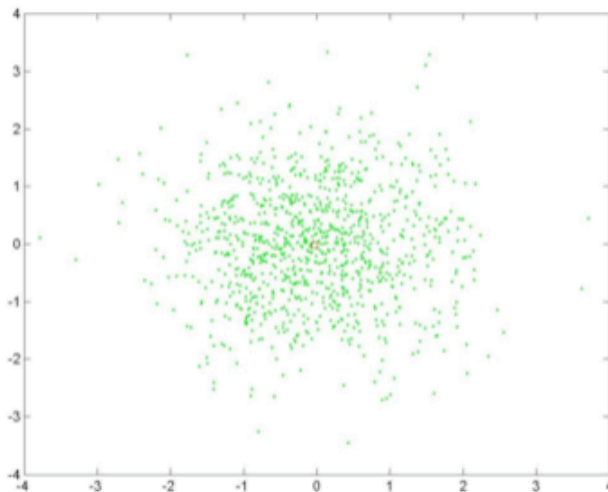
- For uniform distribution :

$$\text{Var}(x) = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

- Standard Deviation:

$$\sigma_x = \sqrt{\text{Var}(x)}$$

Higher Dimensional Random Variables

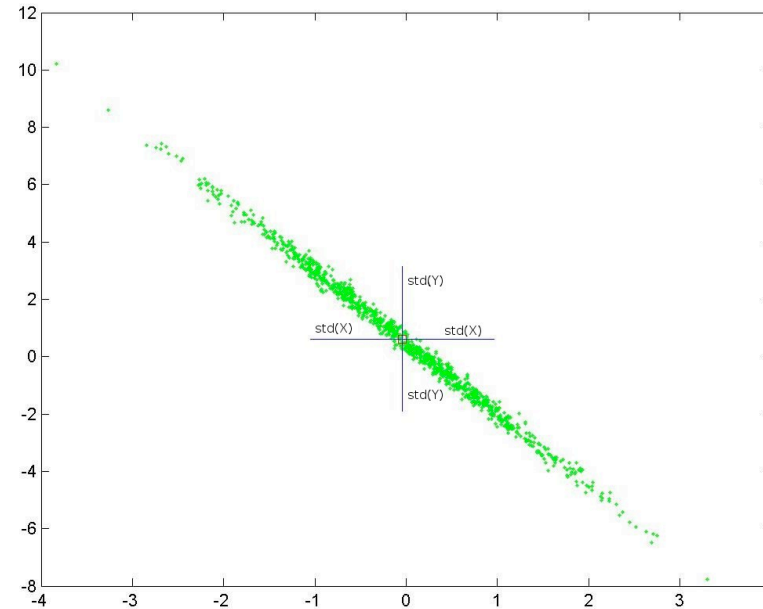
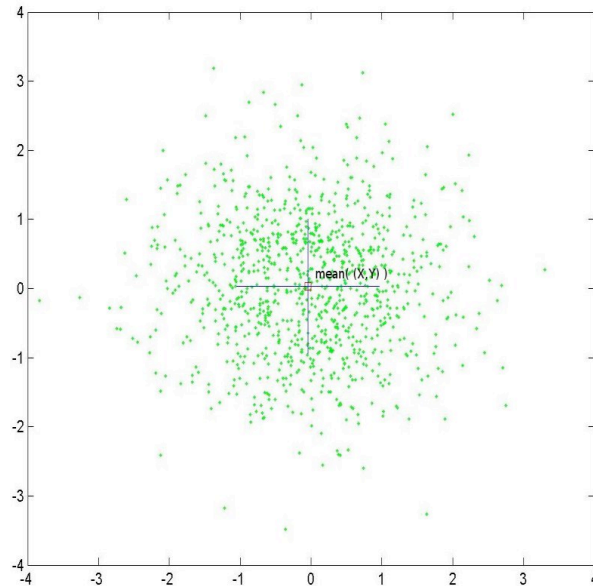


- A higher dimensional random variable $X = (x_1, x_2, \dots, x_n)$ can be interpreted as a set of 1D random variables x_1, x_2, \dots, x_n
- An example for a higher dimensional random variable would be a measurement of a 2D point
 - The measurements in X and Y directions (X, Y) are 1D random variables
 - Together they give the position vector $P = (X, Y)$ as a 2D random variable



Visualizing PDF Properties

- Mean: the centroid of (mass center) of the cloud
- Variance: variation of data in the single directions



Covariances

- Covariance is the generalization of the variance
- We assume uniform distribution
- Let x, y be the components of the higher dimensional random variable $X = (x, y, \dots)$ with the respective means \bar{x}, \bar{y}

$$Var(x) = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(x_i - \bar{x})$$

$$\begin{aligned} Cov(x, y) &= \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) \\ &= E[(x - \bar{x})(y - \bar{y})] \\ &= E[xy] - E[x]E[y] \end{aligned}$$

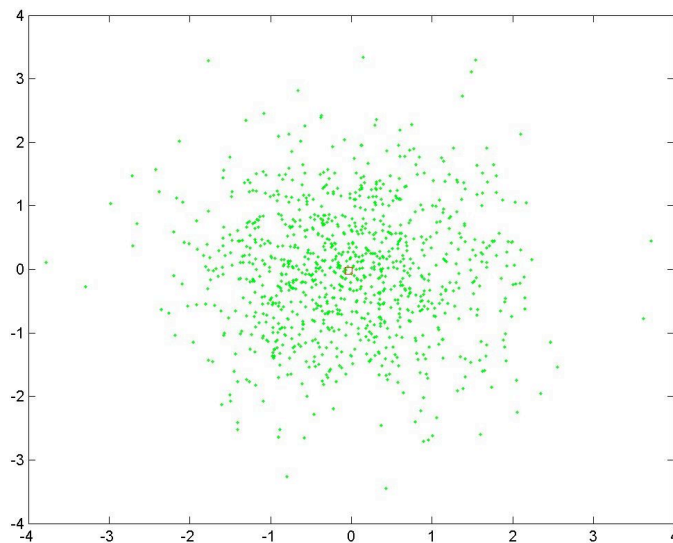
Coariance Matrix

Let $X = (X_1, X_2, \dots, X_n)$ be a higher dimensional random variable.
The corresponding covariance matrix is:

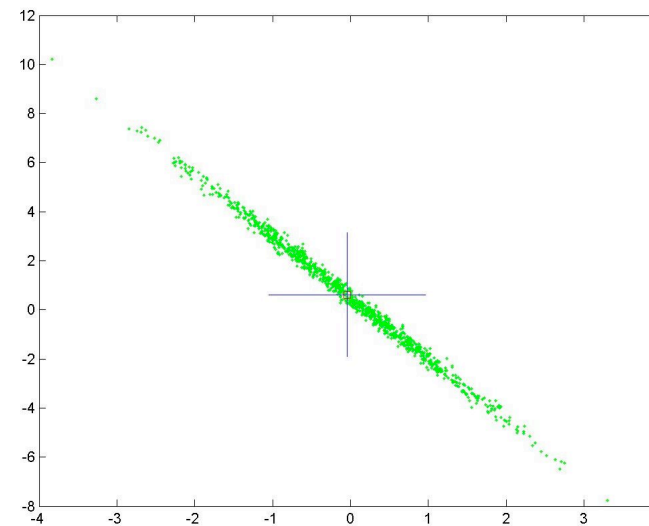
$$C = \begin{bmatrix} Cov(X_1, X_1) & Cov(X_1, X_2) & \dots & Cov(X_1, X_n) \\ Cov(X_2, X_1) & Cov(X_2, X_2) & \dots & Cov(X_2, X_n) \\ \vdots & \vdots & & \vdots \\ Cov(X_n, X_1) & Cov(X_n, X_2) & \dots & Cov(X_n, X_n) \end{bmatrix}$$

Dependency of Variables

- Random variables can be dependent or independent of each other



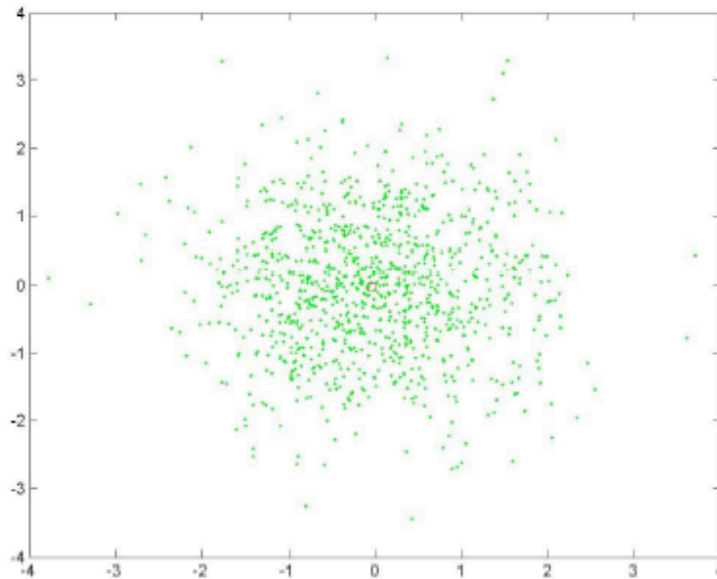
□ X and Y independent



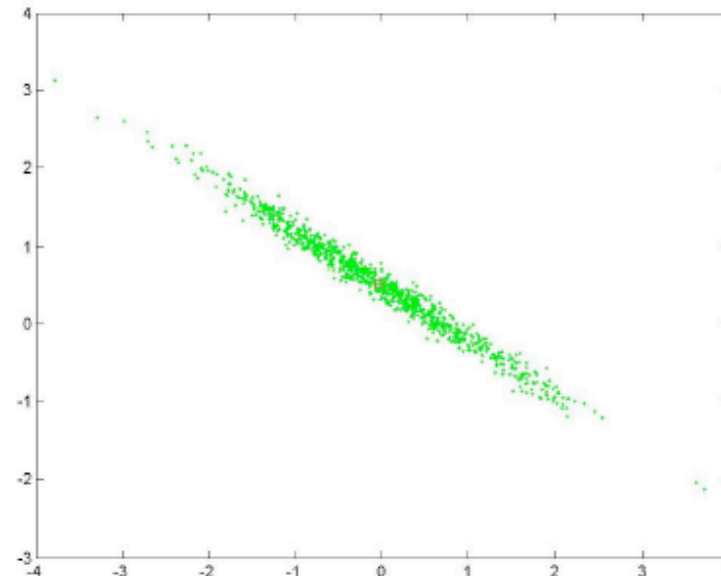
X and Y dependent



Covariance Matrix and Dependency



$$C = \begin{bmatrix} 0.9717 & -0.0069 \\ -0.0069 & 0.9966 \end{bmatrix}$$



$$C = \begin{bmatrix} 0.9717 & -0.6781 \\ -0.6781 & 0.4834 \end{bmatrix}$$



Covariance and Quadrics

- The joint probability density for n-dim vector Δx :

$$p(\Delta x) = (|2\pi|^{n/2} |C_x|^{1/2})^{-1} \exp(-\frac{1}{2} \Delta x^T C_x^{-1} \Delta x)$$

- The surface of constant probability defined by

$$\Delta x^T C_x^{-1} \Delta x = \text{const}$$

- Which looks pretty like a quadric

$$\begin{aligned} Q = Q(\mathbf{x}) &= Q(x, y, z) = \\ &= a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz = \\ &= (xyz) \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{x}^T \mathbf{A} \mathbf{x} \end{aligned}$$

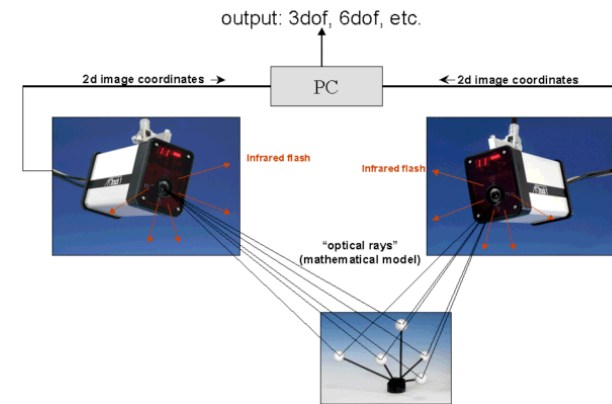
Error in Pose Estimation

- A Pose p has 6DOF:
 - 3DOF for position
 - 3DOF for orientation
- Covariance matrix of pose errors is a 6x6 matrix
- The upperleft 3x3 matrix is the positional part
- The lowerright 3x3 matrix is the rotational part

$$C_x = \begin{pmatrix} Cov(x,x) & Cov(x,y) & Cov(x,z) & Cov(x,\alpha) & Cov(x,\beta) & Cov(x,\gamma) \\ Cov(y,x) & Cov(y,y) & Cov(y,z) & Cov(y,\alpha) & Cov(y,\beta) & Cov(y,\gamma) \\ Cov(z,x) & Cov(z,y) & Cov(z,z) & Cov(z,\alpha) & Cov(z,\beta) & Cov(z,\gamma) \\ Cov(\alpha,x) & Cov(\alpha,y) & Cov(\alpha,z) & Cov(\alpha,\alpha) & Cov(\alpha,\beta) & Cov(\alpha,\gamma) \\ Cov(\beta,x) & Cov(\beta,y) & Cov(\beta,z) & Cov(\beta,\alpha) & Cov(\beta,\beta) & Cov(\beta,\gamma) \\ Cov(\gamma,x) & Cov(\gamma,y) & Cov(\gamma,z) & Cov(\gamma,\alpha) & Cov(\gamma,\beta) & Cov(\gamma,\gamma) \end{pmatrix}$$

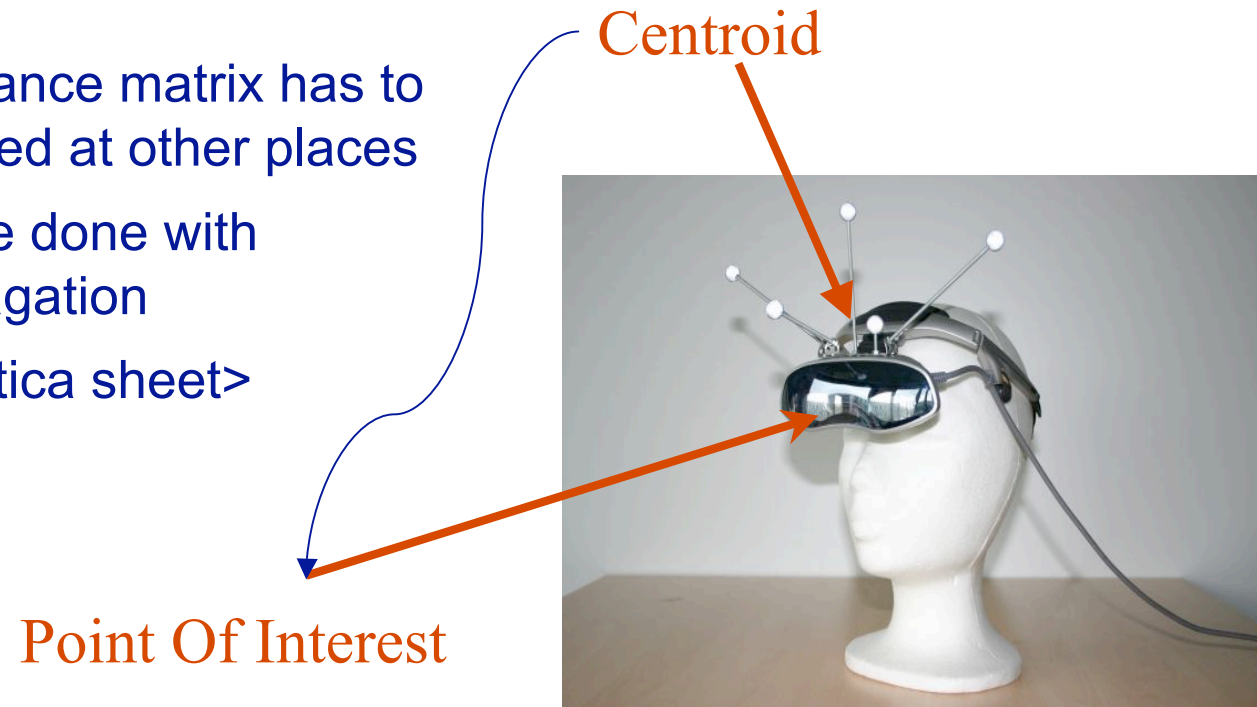
Example of a Tracking System

- Tracking System of A.R.T. GmbH
- Two or more cameras
- Cameras send out IR light
- Reflective marker balls
- System delivers 6DOF data for previous registered targets
- Error Covariance can be calculated



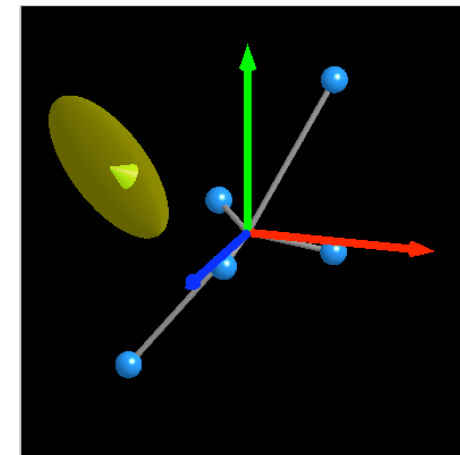
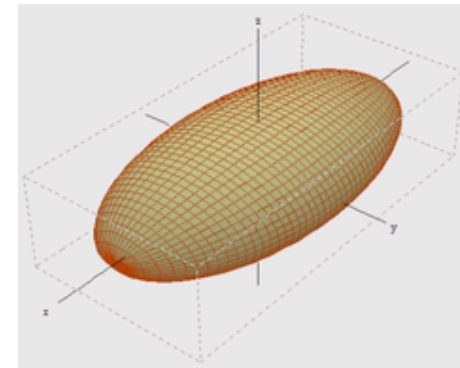
Error Propagation of Pose Errors

- In most cases the covariances in centroid are not interesting
- The covariance matrix has to be calculated at other places
- This can be done with error propagation
- <Mathematica sheet>



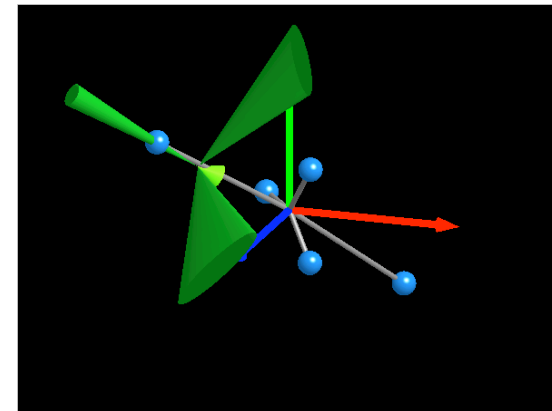
Visualization of Positional Uncertainty

- Surface of const. probability is an ellipsoid
- The covariance matrix can be decomposed into
$$C_x = RDR^T$$
 - With R an orthogonal matrix and D diagonal matrix
 - D consists of Eigenvalues of C_x
 - R consists of Eigenvectors of C_x
- An ellipsoid has three main axes, which are the Eigenvalues of the Covar Matrix C.
- The R rotates the Ellipsoid



Visualize Rotational Uncertainty

- Visualisation of rotational uncertainty can be done with cones
 - This is done by propagating the the rotational 3x3 covariancematrix along an arbitrary axis
 - For each axis we get a „flat“ ellipsoid, perpendicular to that axis
 - In the figure e_1 , e_2 and e_3 was used





Demo