

# 3D Computer Vision - Exercises - WT 2005

## Reminder Products in Vector Spaces

Martin Groher

CAMP - Chair for Computer Aided Medical Procedures

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- 1 Scalar Product
- 2 Vector Product
- 3 Summary and References

# Def

- Given two vectors  $\mathbf{a}$  and  $\mathbf{b}$ , the scalar product is defined as

$$\mathbf{a} \circ \mathbf{b} = \mathbf{a}^T \mathbf{b} = (a_1, a_2, \dots, a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \sum_{i=1}^n a_i b_i$$

# Properties

- Also called *dot product*
- Is commutative ( $\mathbf{a}^T \mathbf{b} = \mathbf{b}^T \mathbf{a}$ )
- Is distributive ( $\mathbf{a}^T (\mathbf{b} + \mathbf{c}) = \mathbf{a}^T \mathbf{b} + \mathbf{a}^T \mathbf{c}$ )
- Is scalar-associative ( $(s\mathbf{a}^T) \mathbf{b} = s(\mathbf{a}^T \mathbf{b})$ )
- The scalar product of two vectors expresses the angle between the vectors

$$\mathbf{a}^T \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \phi$$

- If the two vectors are orthogonal to each other (angle  $90^\circ$ ), the scalar product is zero

$$\mathbf{a}^T \mathbf{b} = 0 \Rightarrow \mathbf{a} \perp \mathbf{b}$$

# Projection

- One can project a vector  $\mathbf{a}$  onto another vector  $\mathbf{u}$  using the scalar product:

$$proj(\mathbf{a}) = \frac{\mathbf{a}^T \mathbf{u}}{\|\mathbf{u}\|^2} \mathbf{u}$$

- Do not get confused: In the lecture we talk about *perspective* projections, this is an *orthogonal* projection!

# Def

- The vector (cross) product is defined on 3-vectors only



$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

- can be developed using a determinant

# Antisymmetric Matrix

- The vector product can also be expressed with the antisymmetric matrix of a vector
- The antisymmetric matrix of vector  $\mathbf{a}$  is

$$[\mathbf{a}]_{\times} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

- The antisymmetric matrix is not only defined for 3-vectors, but it encapsulates the vector-product for them:

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_{\times} \mathbf{b}$$

# Properties

- Is anti-commutative ( $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ )
- Is not associative ( $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ )
- The resulting vector  $\mathbf{c}$  of the cross product of  $\mathbf{a}$  and  $\mathbf{b}$  is orthogonal to  $\mathbf{a}$  and  $\mathbf{b}$ :

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} \Rightarrow (\mathbf{c} \perp \mathbf{a} \wedge \mathbf{c} \perp \mathbf{b})$$

- If two perpendicular vectors are cross-multiplied the resulting basis defines a right-handed or left-handed coordinate system depending on the operand order

# References

- Mathworld. *<http://mathworld.wolfram.com/>*
- Vachenauer. *Höhere Mathematik 1*  
(Chapter ...)