

3D Computer Vision - WT 2005

Higher Dimensional Probability and Statistics Basics

Martin Groher

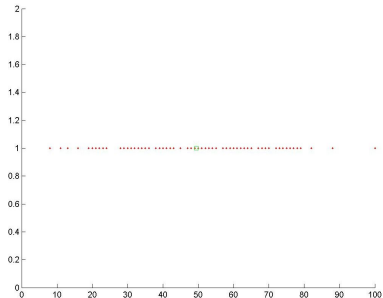
CAMP - Chair for Computer Aided Medical Procedures

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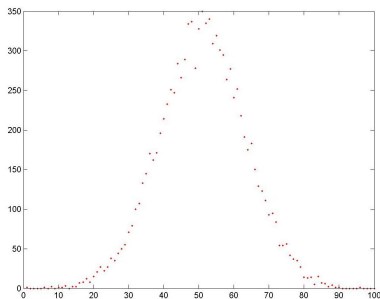
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 - Variance & Standard Deviation
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1D Random Variables



- 1D Random Variable X:
Variable that can take several random values
- CV intuition: set of measurements of one value. Measurement errors are reason for “randomness”

Probability Density Functions

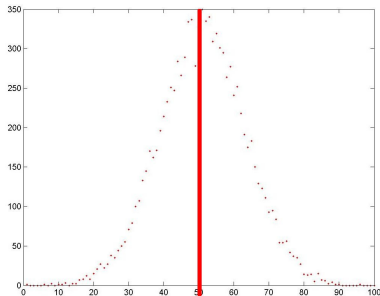


- The PDF $P(X = x)$ expresses the probability that the variable X takes a certain value x
- An example for a PDF is the Gaussian (Normal) PDF:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \cdot \exp\left(-\frac{(x - \bar{x})^2}{2\sigma^2}\right)$$

- The Gaussian will be used quite often in the lecture

Expectation

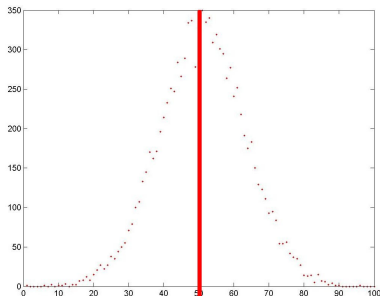


- The expected value for a random variable x :

$$E[x] = \sum_{i=1}^n P(x_i)x_i$$

- Expectation of x is also denoted by $\langle x \rangle$ or μ_x

Mean

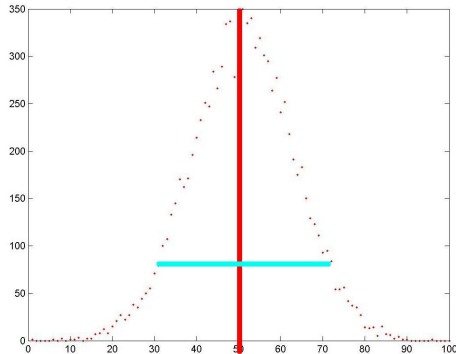


- The mean of a random variable is the average or center of mass:

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

- In CV we often do not know the expectation value of a normal distribution *a priori*.
- E_{Normal} can be approximated (ML approx) via the mean
 $E = \bar{x}$

Variance & Standard Deviation - Intuition



Variance and Standard Deviation basically express how much the values are spread around the expected value

Variance & Standard Deviation

- Variance:

$$\begin{aligned} \text{Var}(x) &= \sum_{i=1}^n P(x_i)(x_i - \bar{x})^2 \\ &= E[(x_i - E[x])^2] \\ &= E[x^2] - E[x]^2 \end{aligned}$$

- The variance of a normal distribution can be approximated (ML approx) via:

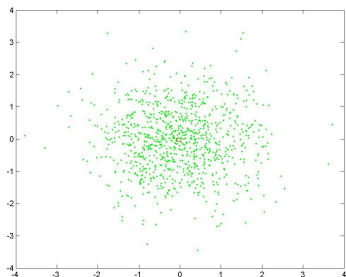
$$\text{Var}(x) = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

- Standard Deviation:

$$\sigma_x = \sqrt{\text{Var}(x)}$$



Higher Dimensional Random Variables



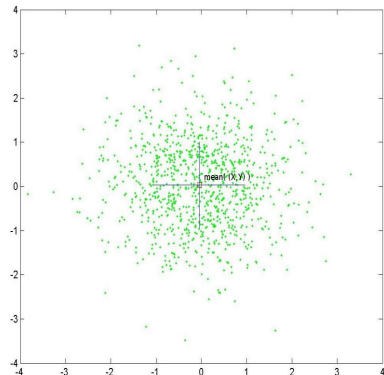
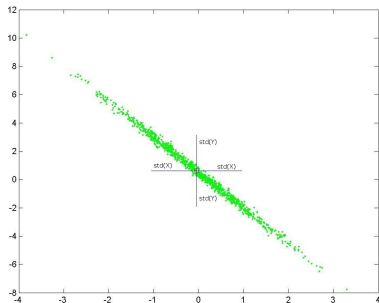
- A higher dimensional random variable $X = (x_1, x_2, \dots, x_n)$ can be interpreted as a set of 1D random variables x_1, x_2, \dots, x_n
- An example for a higher dimensional random variable would be a measurement of a 2D point
 - The measurements in X and Y directions (X, Y) are 1D random variables
 - Together they give the position vector $P = (X, Y)$ as a 2D random variable

Expectation, Mean, Variance and Standard Deviation

Extending PDF's and their properties to higher dimensions is straight forward.

$$E[X] = \begin{bmatrix} E[x_1] \\ E[x_2] \\ \vdots \\ E[x_n] \end{bmatrix}, \text{Var}(X) = \begin{bmatrix} \text{Var}(x_1) \\ \text{Var}(x_2) \\ \vdots \\ \text{Var}(x_n) \end{bmatrix}, \sigma_X = \begin{bmatrix} \sigma_{x_1} \\ \sigma_{x_2} \\ \vdots \\ \sigma_{x_n} \end{bmatrix}$$

Visualizing PDF Properties



- Mean: the centroid (mass center) of the cloud.
- Variances: variation of points in the single directions.

Covariance

- Covariance is the generalization of the variance
- Let x, y be the components of the higher dimensional random variable $X = (x, y, \dots)$ with the respective expectation values $E[x], E[y]$

$$\begin{aligned} \text{Var}(x) &= E[(x_i - E[x])^2] \\ &= E[x^2] - E[x]^2 \\ &= E[xx] - E[x]E[x] \end{aligned}$$

$$\begin{aligned} \text{Cov}(x, y) &= E[(x - E[x])(y - E[y])] \\ &= E[xy] - E[x]E[y] \end{aligned}$$

Covariance Approximation

- Again, the covariance of a normally distributed multivariate random variable can be approximated if the expectation value is not known a priori using the mean

$$Var(x) = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(x_i - \bar{x})$$

$$Cov(x, y) = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$$

Covariance Matrix

Let $X = (X_1, X_2, \dots, X_n)$ be a higher dimensional random variable.
The corresponding covariance matrix is:

$$C = \begin{bmatrix} Cov(X_1, X_1) & Cov(X_1, X_2) & \dots & Cov(X_1, X_n) \\ Cov(X_2, X_1) & Cov(X_2, X_2) & \dots & Cov(X_2, X_n) \\ \vdots & \vdots & & \vdots \\ Cov(X_n, X_1) & Cov(X_n, X_2) & \dots & Cov(X_n, X_n) \end{bmatrix}$$

Dependency of Variables

Random variables can be dependent or independent of each other

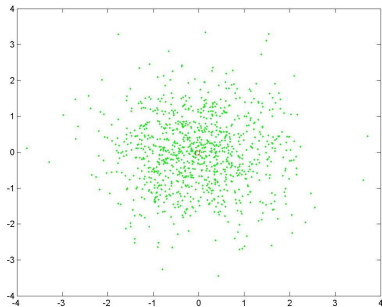


Figure: X and Y components independent

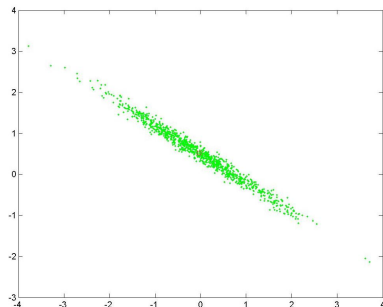
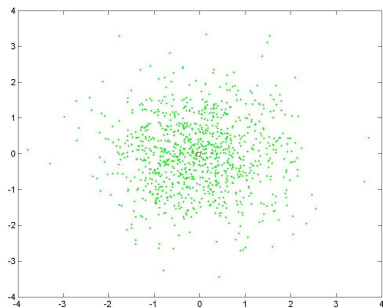


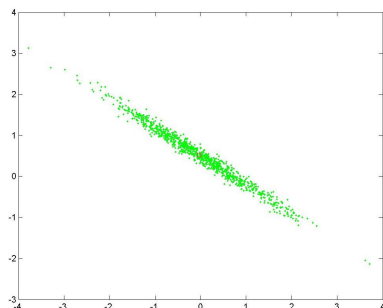
Figure: X and Y components dependent

Covariance Tells You about Dependency of Variables

Covariance of two variables tells you whether they are correlated



$$C = \begin{bmatrix} 0.9717 & -0.0069 \\ -0.0069 & 0.9966 \end{bmatrix}$$



$$C = \begin{bmatrix} 0.9717 & -0.6781 \\ -0.6781 & 0.4834 \end{bmatrix}$$



Covariance and Dependency of Variables - How come...?

- Assume X and Y independent

$$\begin{aligned}Cov(X, Y) &= E[XY] - E[X]E[Y] \\ &= E[X]E[Y] - E[X]E[Y] \\ &= 0\end{aligned}$$

- if $Cov(X, Y) > 0$ then X and Y are dependent s.t. if Y increases with increasing X
- if $Cov(X, Y) < 0$ then X and Y are dependent s.t. if Y decreases if Y increases

Covariance Applications in CV

- Mahalanobis Distance:

$$\|X - \bar{X}\|_{\Sigma}^2 = (X - \bar{X})^T \Sigma^{-1} (X - \bar{X})$$

- Error Propagation: the covariance matrix captures the uncertainty of the computed transformation.

Correlation

- Correlation is a slight modification of the covariance
- Given random variables x, y with means \bar{x}, \bar{y} and standard deviations σ_x, σ_y

- Cov:

$$Cov(x, y) = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$$

- Cor:

$$Cor(x, y) = \frac{Cov(x, y)}{\sigma_x \sigma_y} = \frac{1}{n} \sum_{i=1}^n \frac{(x_i - \bar{x})(y_i - \bar{y})}{\sigma_x \sigma_y}$$

- We can build a **Correlation Matrix** in the same way as the covariance matrix
- Application of Correlation as similarity measure in CV



Summary

Reminder 1D Probability:

- Random Variables
- Probability Density Functions (PDF)
- Expectation and Mean
- Variance and Standard Deviation

Multivariate Probability:

- Higher Dimensional Random Variables
- Expectation and Mean
- Variance and Standard Deviation
- Covariance
- Covariance as Dependency Measure
- Correlation

MatLab Commands

- $X = \text{randn}([m \ n])$ creates an m by n matrix X with normally distributed random entries
Can be used to generate a higher dimensional random variable (e.g. m -dimensional variable with n measurements).
- $X = \text{rand}([m \ n])$ creates uniformly distributed entries within range 0-1.
- $\text{mean}(X)$ Mean of the random variable X
- $\text{var}(X)$ Variance of the random variable X
- $\text{std}(X)$ Standard deviation of the random variable X
- $\text{cov}(X)$ Covariance of the random variable X

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