

Exercises in 3D Computer Vision

Exercise 30 Self-calibration of the distortion of a zooming camera

This exercise presents a new method for the self-calibration of the lens distortion of a zooming camera which appears for short focal lengths. The proposed technique does not need any special calibration pattern nor any prior knowledge about the environment. The key idea is to match points between a distorted image and an undistorted image taken at different resolutions. Matched points are used to compute invariants to the pinhole camera parameters. Then, lens distortion parameters are estimated in order to obtain the same invariants in both images. This approach is well suited to autonomous robotic vision applications. In fact, the self-calibration of the camera is done before moving the robot.

Let (X_1, X_2, \dots, X_n) be 3D points belonging to a certain object, where (X_j, Y_j, Z_j) are the coordinates of X_j expressed in a absolute frame $(C, \vec{i}, \vec{j}, \vec{k})$. Suppose that we have a zooming camera observing the object. Let \mathbf{K}_i be the camera intrinsic parameters at the moment of the capture of the image i :

$$\mathbf{K}_i = \begin{bmatrix} f_i & 0 & u_{0i} \\ 0 & f_i & v_{0i} \\ 0 & 0 & 1 \end{bmatrix}$$

Let $\mathbf{m}_j = [x_j, y_j, 1]^T$ be the projection of the 3D point X_j on the plane π perpendicular to the optical axis \vec{k} and verifying: $distance(C, \pi) = 1$, and let $\mathbf{p}_{ji} = [u_{ji}, v_{ji}, 1]^T$ be its projection in the image i (u_{ji} and v_{ji} are in pixels).

- a) Give the expression of x_j, y_j, u_{ji} and v_{ji} .

$$x_j = \frac{X_j}{Z_j}, y_j = \frac{Y_j}{Z_j} \text{ and } \mathbf{p}_{ji} = \mathbf{K}_i \mathbf{m}_j$$

- b) Let $\mathbf{S}_{pi} = \frac{1}{n} \sum_{j=1}^n \mathbf{p}_{ji} \mathbf{p}_{ji}^T$ and $\mathbf{S}_m = \frac{1}{n} \sum_{j=1}^n \mathbf{m}_j \mathbf{m}_j^T$. Express \mathbf{S}_{pi} and \mathbf{S}_m using x_j, y_j, u_{ji} and v_{ji} .

$$\mathbf{S}_{pi} = \frac{1}{n} \sum_{j=1}^n \mathbf{p}_{ji} \mathbf{p}_{ji}^T = \frac{1}{n} \sum_{j=1}^n \begin{bmatrix} u_{ji}^2 & u_{ji}v_{ji} & u_{ji} \\ u_{ji}v_{ji} & v_{ji}^2 & v_{ji} \\ u_{ji} & v_{ji} & 1 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & 1 \end{bmatrix} \quad (1)$$

$$\mathbf{S}_m = \frac{1}{n} \sum_{j=1}^n \mathbf{m}_j \mathbf{m}_j^T = \frac{1}{n} \sum_{j=1}^n \begin{bmatrix} x_j^2 & x_j y_j & x_j \\ x_j y_j & y_j^2 & y_j \\ x_j & y_j & 1 \end{bmatrix} \quad (2)$$

What is the geometric meaning of the entries of the matrices \mathbf{S}_{pi} and \mathbf{S}_m .

This matrix has a simple geometric meaning. Indeed, σ_{13} and σ_{23} are the coordinates of the centroid of the n points, while σ_{11} , σ_{12} and σ_{22} are the second order moments of the set of points. It is important to remark that the matrix \mathbf{S}_{pi} can be measured using the image points and it is camera dependent while the matrix \mathbf{S}_m can not be measured without a camera calibration and it is camera independent.

c) Give the equation that links \mathbf{S}_{pi} and \mathbf{S}_m .

$$\mathbf{S}_{pi} = \frac{1}{n} \sum_{j=1}^n \mathbf{p}_{ji} \mathbf{p}_{ji}^\top = \frac{1}{n} \sum_{j=1}^n \mathbf{K}_i \mathbf{m}_j \mathbf{m}_j^\top \mathbf{K}_i^\top = \mathbf{K}_i \left(\frac{1}{n} \sum_{j=1}^n \mathbf{m}_j \mathbf{m}_j^\top \right) \mathbf{K}_i^\top = \mathbf{K}_i \mathbf{S}_m \mathbf{K}_i^\top$$

d) Show that \mathbf{S}_{pi} and \mathbf{S}_m are not definite positive for $n < 3$.

For $n = 1$, we have $ad - b^2 = 0$ and for $n = 2$, we have $\det(\mathbf{S}_x) = 0$ ($x = p$ or $x = m$).

e) Let $\mathbf{x} \in \mathbb{R}^n$. Show that we have:

$$\mathbf{x}^\top \mathbf{x} = 0 \text{ iff } \mathbf{x} = \mathbf{0}_{n \times 1}$$

For $n > 3$ non-colinear points, demonstrate that \mathbf{S}_p and \mathbf{S}_m are definite positive matrices.

We can write $\mathbf{S}_p = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \dots & \mathbf{p}_n \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \dots & \mathbf{p}_n \end{bmatrix}^\top = \mathbf{M} \mathbf{M}^\top$.

Then $\forall \mathbf{x} \in \mathbb{R}^3 \neq \mathbf{0}_{n \times 1}$, $\mathbf{x}^\top \mathbf{S}_p \mathbf{x} = \|\mathbf{M}^\top \mathbf{x}\|^2 \geq 0$

We have $\text{rank}(\mathbf{M}^\top) = \text{rank}(\mathbf{M}) \leq \min(3, n)$ because \mathbf{M} is a $(3 \times n)$ matrix.

$\mathbf{M}^\top \mathbf{x} \neq \mathbf{0}_{n \times 1}$ iff $n \geq 3$ and \mathbf{M} has 3 columns linearly independant (i.e. 3 non-colinear points).

Then, for $n \geq 3$ non-colinear points, we have \mathbf{S}_{pi} and \mathbf{S}_m are definite positive matrices.

f) Let \mathbf{T} be the following matrix:

$$\mathbf{T} = \begin{bmatrix} \sqrt{\sigma_{11} - \sigma_{13}^2 - \frac{(\sigma_{12} - \sigma_{13}\sigma_{23})^2}{\sigma_{22} - \sigma_{23}^2}} & \frac{\sigma_{12} - \sigma_{13}\sigma_{23}}{\sqrt{\sigma_{22} - \sigma_{23}^2}} & \sigma_{13} \\ 0 & \sqrt{\sigma_{22} - \sigma_{23}^2} & \sigma_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Compute the product $\mathbf{T} \mathbf{T}^\top$.

$$\mathbf{T} \mathbf{T}^\top = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & 1 \end{bmatrix}$$

Find the expression of the matrices \mathbf{T}_{pi} and \mathbf{T}_m verifying $\mathbf{S}_{pi} = \mathbf{T}_{pi} \mathbf{T}_{pi}^\top$ and $\mathbf{S}_m = \mathbf{T}_m \mathbf{T}_m^\top$.

This is the Cholesky decomposition of the symmetric positive definite matrix. Thanks to the uniqueness of this decomposition the matrices \mathbf{T}_{pi} and \mathbf{T}_m have the same expression as \mathbf{T} with replacing the σ_{xx} with the entries of \mathbf{S}_{pi} and \mathbf{S}_m .

g) Give the relationship between \mathbf{T}_m and \mathbf{T}_{pi} .

We have $\mathbf{S}_p = \mathbf{T}_{pi} \mathbf{T}_{pi}^\top = \mathbf{K}_i \mathbf{S}_m \mathbf{K}_i^\top = \mathbf{K}_i \mathbf{T}_m \mathbf{T}_m^\top \mathbf{K}_i^\top = (\mathbf{K}_i \mathbf{T}_m) (\mathbf{K}_i \mathbf{T}_m)^\top$. Another time thanks to the uniqueness of the Cholesky decomposition the matrices, we have: $\mathbf{T}_{pi} = \mathbf{K}_i \mathbf{T}_m$.

h) Show that the vectors $\mathbf{q}_{ji} = \mathbf{T}_{pi}^{-1} \mathbf{p}_{ji}$ are camera independent.

$$\mathbf{q}_{ji} = \mathbf{T}_{pi}^{-1} \mathbf{p}_{ji} = \mathbf{T}_m^{-1} \mathbf{K}_i^{-1} \mathbf{K}_i \mathbf{m}_j = \mathbf{T}_m^{-1} \mathbf{m}_j$$

We have \mathbf{T}_m and \mathbf{m}_j are camera independent, then \mathbf{q}_{ji} are camera independent.

i) We consider the standard radial distortion model, but any other parametric distortion model could be used. Let $\mathbf{p}_{ci} = (u_{ci}, v_{ci}, 1)$ be the unknown center of radial distortion of the image

i (not necessarily equal to the principal point), and let ρ_{ji} be the distance of a distorted point $\mathbf{p}_{ji}^d = (u_{ji}^d, v_{ji}^d, 1)$ from \mathbf{p}_{ci} :

$$\rho_{ji} = \|\mathbf{p}_{ji}^d - \mathbf{p}_{ci}\| = \sqrt{(u_{ji}^d - u_{ci})^2 + (v_{ji}^d - v_{ci})^2}$$

The undistorted coordinates \mathbf{p}_{ji} of an image point can be computed as:

$$\mathbf{p}_{ji} = \mathbf{p}_{ji}^d + (\mathbf{p}_{ji}^d - \mathbf{p}_{ci})(d_{1i}\rho_{ji}^2 + d_{2i}\rho_{ji}^4 + \dots)$$

The camera distortion parameters of image i are the center of radial distortion \mathbf{p}_{ci} and the vector $\mathbf{d}_i = (d_{1i}, d_{2i}, \dots)$.

Give a method that makes it possible to undistort short focal length images.

If the projection model is perfectly perspective, the invariants computed in the first and the second images should be exactly the same $\mathbf{q}_{j2} = \mathbf{q}_{j1}, \forall j$. Due to image distortion in one of the images, there is a difference between the invariants. We suppose that the lens distortion of a zooming camera appears only for short focal lengths and can be neglected for long focal lengths. We will consider that only one image is distorted. The undistorted image coordinates can be computed from a parametric model of the distortion. Let \mathbf{p}_{c1} and \mathbf{d}_1 be respectively the center and the coefficients of distortion for the image 1. The undistorted points in image 1 are a function of \mathbf{p}_{c1} and \mathbf{d}_1 :

$$\mathbf{p}_{j1} = f(\mathbf{p}_{j1}^d, \mathbf{p}_{c1}, \mathbf{d}_1)$$

After computing the matrices \mathbf{T}_{p1} and \mathbf{T}_{p2} , we obtain the invariants to pinhole camera parameters:

$$\mathbf{q}_{j1} = \mathbf{T}_{p1}^{-1} \mathbf{p}_{j1} = \mathbf{q}_{j1}(\mathbf{p}_{c1}, \mathbf{d}_1)$$

and:

$$\mathbf{q}_{j2} = \mathbf{T}_{p2}^{-1} \mathbf{p}_{j2}$$

Thus, we can define a cost function which depends on the difference between the invariants computed from the two images:

$$\mathcal{C}(\mathbf{p}_{c1}, \mathbf{d}_1) = \sum_{j=1}^n \|\mathbf{q}_{j2} - \mathbf{q}_{j1}(\mathbf{p}_{c1}, \mathbf{d}_1)\|^2$$

The distortion parameters can be found with a nonlinear optimization method by minimizing the cost function:

$$\min \mathcal{C}(\mathbf{p}_{c1}, \mathbf{d}_1)$$