

Singular Value Decomposition

Basic Mathematical Tools for Imaging and Visualization (WT 2006)

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Singular Value Decomposition (SVD)

- For any given matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ there exists a decomposition

$$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^T$$

such that

- \mathbf{U} is an $(m \times m)$ orthogonal matrix
- \mathbf{S} is an $(m \times n)$ matrix with non-negative diagonal entries
- \mathbf{V} is an $(n \times n)$ orthogonal matrix

SVD - Visualized

$$\underbrace{\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}}_{\mathbf{A}(m \times n)} = \underbrace{\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}}_{\mathbf{U}(m \times m)} \underbrace{\begin{bmatrix} \sigma_1 & & & & \\ & \cdot & & & \\ & & \sigma_n & & \\ 0 & \cdot & 0 & & \\ 0 & \cdot & 0 & & \end{bmatrix}}_{\mathbf{S}(m \times n)} \underbrace{\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}}_{\mathbf{V}^T(n \times n)}$$

- \mathbf{A} is a $(m \times n)$ matrix
- \mathbf{U} is a $(m \times m)$ orthogonal matrix
- \mathbf{S} is a $(m \times n)$ matrix with non-negative diagonal entries
- \mathbf{V} is a $(n \times n)$ orthogonal matrix

Singular Value Decomposition (SVD)

- The diagonal values of \mathbf{S} are called **Singular Values** of \mathbf{A}
- The column vectors of \mathbf{U} are the **Left Singular Vectors** of \mathbf{A}
- The column vectors of \mathbf{V} are the **Right Singular Vectors** of \mathbf{A} .

Properties

- The SVD can be performed s.t. the diagonal values of \mathbf{S} are descending i.e. $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$.
- We will assume that the SVD is always performed in that way
- The diagonal values of \mathbf{S} are the square roots of the Eigenvalues of $\mathbf{A}^\top \mathbf{A}$ and $\mathbf{A} \mathbf{A}^\top$
(Hence the non-negativity of the entries of \mathbf{S})

Some More Properties

- It holds for the left singular vectors \mathbf{u}_i :

$$\mathbf{A}^\top \mathbf{A} \mathbf{u}_i = \sigma_i^2 \mathbf{u}_i$$

- It holds for the right singular vectors \mathbf{v}_i :

$$\mathbf{A} \mathbf{A}^\top \mathbf{v}_i = \sigma_i^2 \mathbf{v}_i$$

- The left singular vectors \mathbf{u}_i are eigenvectors of $\mathbf{A}^\top \mathbf{A}$
- The right singular vectors \mathbf{v}_i are eigenvectors of $\mathbf{A} \mathbf{A}^\top$

Even More Properties

- SVD explicitly constructs orthonormal bases for the null-space and the range of a matrix
- columns of \mathbf{U} corresponding to non-zero entries of \mathbf{S} span the range
- columns of \mathbf{V} corresponding to zero entries of \mathbf{S} span the null-space

Properties Galore

- SVD allows a rank decision:

$rank(\mathbf{A})$ is the largest r s.t. $\sigma_r > 0$

- there are $m - r$ left singular vectors corresponding to the singular value 0.
- there are $n - r$ right singular vectors corresponding to the singular value 0.

Linear Optimization

SVD can be used for linear optimization by using the following property

- Let \mathbf{v}_n be the right singular vector corresponding to σ_n (the smallest entry of \mathbf{S})
- The product $\mathbf{A}\mathbf{x}$ with $\|\mathbf{x}\|_2 = 1$ has the minimal value for $\mathbf{x} = \mathbf{v}_n$.

Minimization by SVD

The minimizing property of the last right singular vector \mathbf{v}_n can be used to solve the following minimization task

- Given the linear function $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$, $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ to be minimized (in most applications $m \gg n$)
- With the constraint that the solution \mathbf{x} is not trivial ($\mathbf{x} \neq \mathbf{0}$)
(We will assume that $\|\mathbf{x}\|_2 = 1$)

Minimization by SVD II

- The minimization problem is thus

$$\text{minimize } \mathbf{Ax} \quad \text{s.t. } \|\mathbf{x}\|_2 = 1$$

- It can be shown that the solution is the right vector $\mathbf{x} = \mathbf{v}_n$ corresponding to the smallest singular value σ_n .

Proof

- Problem: minimize $\|\mathbf{Ax}\|_2$ subject to $\|\mathbf{x}\|_2 = 1$
- Because of the orthogonality of \mathbf{U} and \mathbf{V} we have
 - $\|\mathbf{Ax}\|_2 = \|\mathbf{USV}^\top \mathbf{x}\|_2 = \|\mathbf{SV}^\top \mathbf{x}\|_2$
 - $\|\mathbf{x}\|_2 = \|\mathbf{V}^\top \mathbf{x}\|_2$
- Hence have to minimize $\|\mathbf{SV}^\top \mathbf{x}\|_2$ subject to $\|\mathbf{V}^\top \mathbf{x}\|_2 = 1$
- With $\mathbf{V}^\top \mathbf{x} = \mathbf{y}$ we have: minimize $\|\mathbf{Sy}\|_2$ subject to $\|\mathbf{y}\|_2 = 1$
- Since \mathbf{S} diagonal with descending entries we get $\mathbf{y} = [0, 0, \dots, 0, 1]^\top$
- Since $\mathbf{V}^\top \mathbf{x} = \mathbf{y} \Leftrightarrow \mathbf{x} = \mathbf{Vy}$ we get $\mathbf{x} = \mathbf{v}_n$

What's next...?

So in order to solve a linear minimization problem by SVD we have to do two things

- 1 State it in the form

$$\text{minimize } \mathbf{Ax} \quad \text{s.t.} \quad \|\mathbf{x}\|_2 = 1$$

- 2 Compute the SVD $\mathbf{A} = \mathbf{USV}^\top$ and take the last right vector \mathbf{v}_n as the solution

Notes

- Minimization by SVD is extremely simple. The tricky part is to state the problem the right way.
- The error minimized by the SVD is called **Algebraic Error** or **Algebraic Distance**

SVD Black Box

- Only thing left is to show how an SVD of a matrix \mathbf{A} can be computed
- But we won't do that!
- We'll use SVD as a black box
- Computation involves QR procedure and Householder reduction
- Original algorithm by Golub and Reinsch

Computation Properties

- Algorithm is extremely stable
- Computation time for SVD of a $(m \times n)$ matrix \mathbf{A} :
 - Computation of \mathbf{U} , \mathbf{V} and \mathbf{S} : $4m^2n + 8mn^2 + 9n^3$
 - Computation of \mathbf{V} and \mathbf{S} : $4mn^2 + 8n^3$
- Keep in mind that in most cases $m \gg n$

Implementation in Matlab

- Matlab has the command $[U,S,V] = \text{SVD}(X)$
- Attention: The command returns V and not V^T
- Hence it holds that $X = USV^T$

Summary

- Properties of the SVD
- Linear minimization using SVD

References

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