

Basic Mathematical Tools for Imaging and Visualization

Minimization of Functionals Basics and Applications

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- **Until now**, we minimized functions

$$f : \mathbb{R}^n \rightarrow \mathbb{R}$$

and looking for a finite number of parameters (n-dimensional vector) for which the function f takes the minimal value

- **Now**, we want to minimize functionals

$$J[u] = \int F[u, \cdot] dx$$

where the unknown u is not a finite-dimensional vector but a function

Why the heck should we do something like that?

1. Suffering is good for you...
2. There are some applications around

Exemplary Applications

- Image Inpainting
- Image Deconvolution
- Segmentation
- Deformable Registration

- Still Why?
 - Continuous vs. Discrete Modelling
 - Powerful Tools from Analysis can be used in continuous case

Image Deconvolution



Recorded (Blurred)
Image

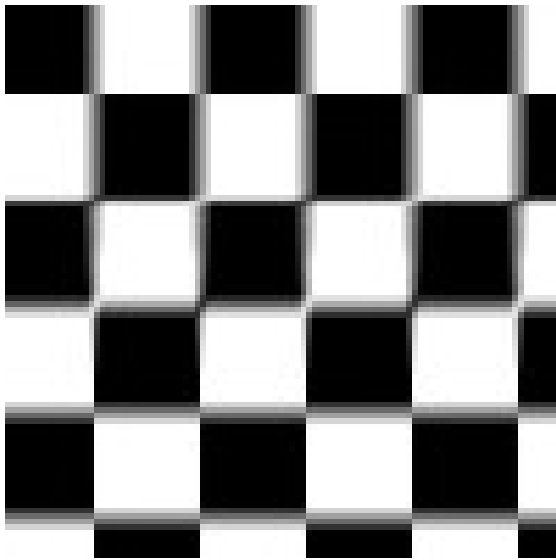


Reconstructed Image
(DECONVOLUTION)

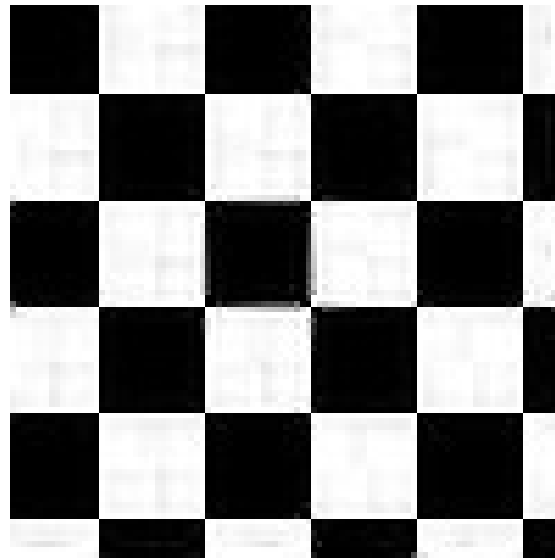


'True' Image

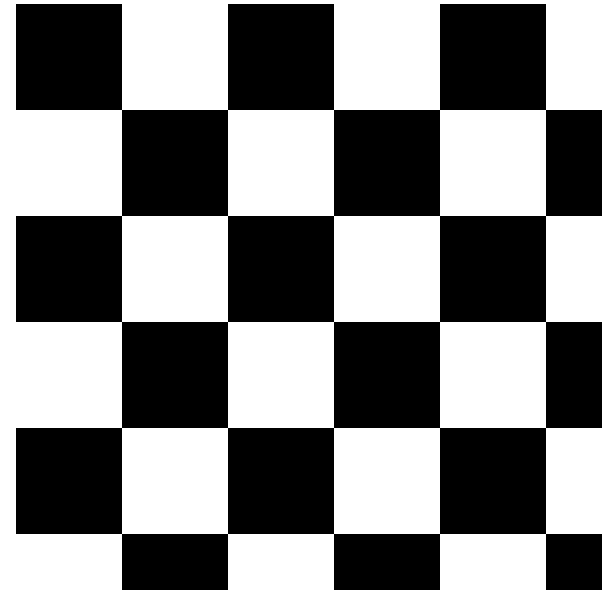
Image Deconvolution



Recorded (Blurred)
Image

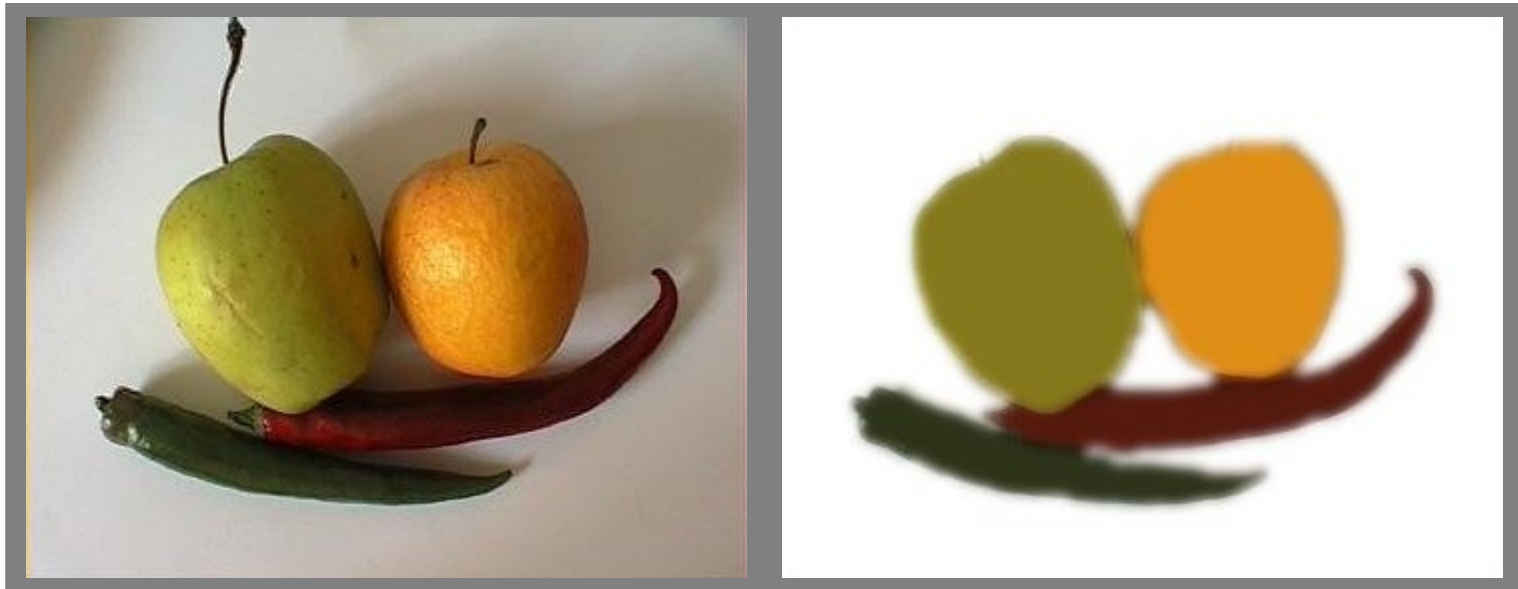


Reconstructed Image
(DECONVOLUTION)

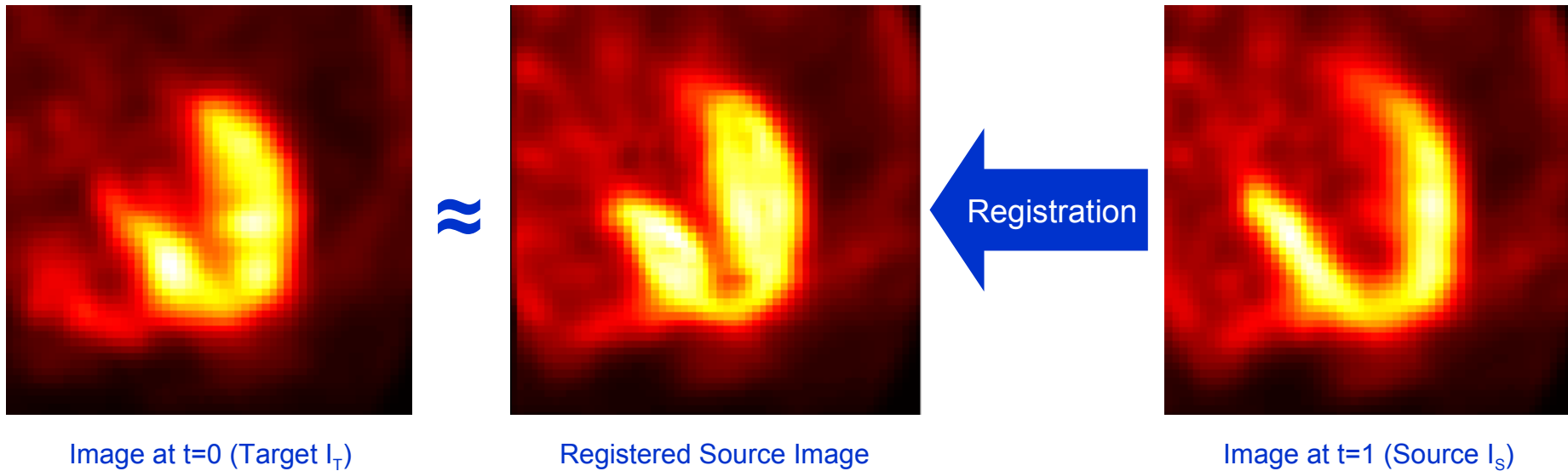


'True' Image

Segmentation



Deformable Registration



Deformable Registration

Undo deformation of an image, such that it resembles the previous image

Image Inpainting

All examples from webpage og Guillermo Sapiro
(<http://www-mount.ee.umn.edu/~guille/inpainting.htm>)





'General' Optimization Approach

- Assign certain value/cost to every possible configuration such that this cost is minimal for the 'true' configuration
- Value = Cost = "Error"

- However, measuring only the Error gives bad algorithms

→ ILL-POSEDNESS

- Remedy: **Regularization**

Well-Posedness (Hadamard):

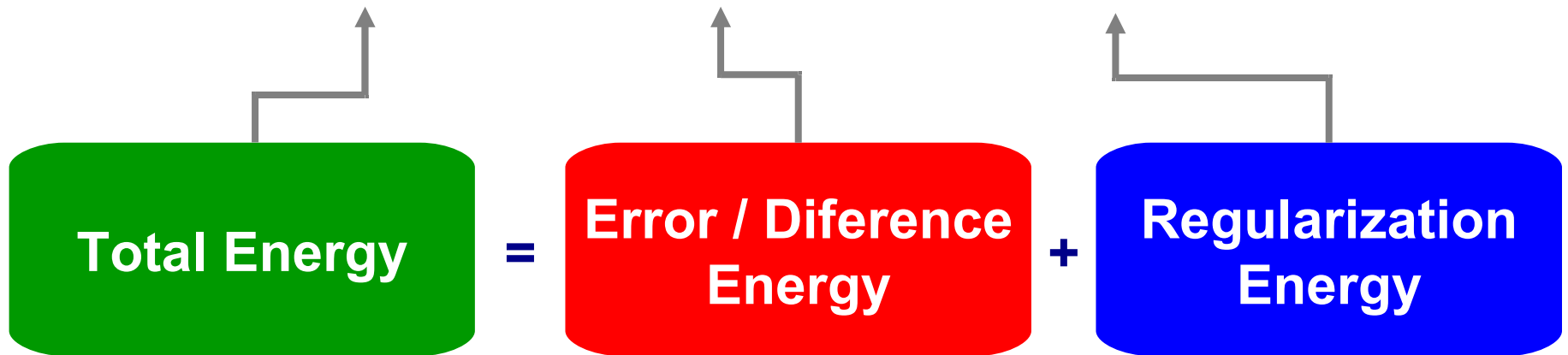
- a solution exists
- solution is unique
- problem is well-conditioned:
 - = errors in output depend continuously on errors in input
 - = small input errors do not cause large errors in result

Regularization

- Making the problem mathematically well-posed
- Encoding a-priori knowledge about the problem

Typical Image Processing Problem

$$J = D + \alpha S$$



Outline

1. Some examples for the modelling on the Functional Level
2. Derivation of the Euler-Lagrange Equations for a standard model problem
 - Common flavors of the model problem
 - Examples
- Case-Study: Deformable Registration

Modelling

- Some Functionals you already know from lecture?
- Typical Regularization Methods

Some Model Examples

$$J[u] = \int_{\Omega} (I_1(x) - I_2(u(x)))^2 dx$$

- The good old SSD

$$J[u] = \int_{\Omega} \|u_0(x) - u(x)\|^2 dx$$

- Measure Difference to u_0 :

- L2 norm

$$J[u] = \int_{\Omega} \|u_0(x) - u(x)\| dx$$

- L1 norm

$$J[u] = \int_{\Omega} \|u_0(x) - u(x)\| \cdot \|u_0(x)\| dx$$

- L1 norm, weighted

More Model Examples

$$J[u] = \int_{\Omega} \|u(x)\|^2 dx$$

- Keep the magnitude of u small

$$J[u] = \int_{\Omega} \|\nabla u(x)\|^2 dx$$

- Keep magnitude of gradient of u small
→ Impose smoothness on u

$$J[u] = \int_{\Omega} \|\nabla u(x)\| dx$$

$$J[u] = \int_{\Omega} \|\Delta u(x)\|^2 dx$$

- Keep magnitude of laplace of u small
→ Impose smoothness on u
→ possibly smooth curvature*

$$J[u] = \int_{\Omega} \|\Delta u(x)\| dx$$

*(only an approximation for dimensions > 1)

Computing the Solution to the Model

- Given a functional F
- Calculus of Variations provides us with the Euler-Lagrange Equation as necessary condition for an extremal of F

Functional

corresponding Euler-Lagrange Equation

$$\min_u \int_{\Omega} (I_1(x) - I_2(u(x)))^2 dx \quad \Rightarrow \quad 0 = -2 \cdot (I_1(x) - I_2(u(x))) \nabla I_2(u(x))$$

$$\min_u \int_{\Omega} \|\nabla u(x)\|^2 dx \quad \Rightarrow \quad 0 = \Delta u(x)$$

$$\min_u \int_{\Omega} \|\Delta u(x)\|^2 dx \quad \Rightarrow \quad 0 = \Delta^2 u(x)$$

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Problem Model as Functional

Modelling on *Variational Level*
("Energy Level")

Euler-Lagrange Differential Equation
(non-linear)

Necessary Condition for Minimum of Functional
("Don't minimize a functional – solve an equation!")

Quasilinearization

Approximate the solution of the non-linear problem by
an iteration using a linearized version of the problem
(Fixed-Point Iteration, Time Marching, ...)

Discretization

Discretize the linear problem in order to actually
compute the solution on the computer
(Finite Difference Method, Finite Element Method...)

Solve Linear System

Methods are needed for solving the linear systems
arising in the Quasilinearization
(Direct Solvers, Iterative Methods (Conjugate Gradients, Jacobi, Gauß-
Seidel, SOR, Multigrid))

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