

3D Computer Vision Camera Models

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based on a course given at UNC by Marc Pollefeys & the book "Multiple View Geometry" by Hartley & Zisserman

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Outline of the Computer Vision Lecture

Projective Geometry and Transformations

- Transformations with homogeneous coordinates
- 2D projective geometry
- 3D projective geometry
- 1D projective geometry

Estimation

- Direct and Iterative Estimation
- DLT Algorithm
- Cost Functions
- Robust Estimation (RANSAC)

Camera Geometry

- Camera Models
- Camera Calibration

Algorithm Evaluation and Error Analysis

- Statistics, Error Propagation

Advanced Estimation

- Bundle Adjustment
- Iterative Closest Point (ICP)



Single-View Geometry

Three-View Geometry

N-View Geometry

Two-View Geometry

- Epipolar Geometry / Fundamental Matrix / Essential Matrix
- 3D Reconstruction of Cameras and Structure

- Trifocal Tensor
- Computation of the Trifocal Tensor

- N-Linearities and Multiple View Tensors
- N-View Computational Methods
- Factorization
- Structure From Motion

Outline – Camera Model

- History of Camera (Camera Obscura)
- Intensity Images
- Geometric Parameters of a Finite camera
- Projective Camera
 - Camera Center, Principal Plane, Axis Plane, Principal Point, Principal Ray
 - Decomposition of Camera Matrix
- Cameras at Infinity
- Other Camera Models (Pushbroom and Line Cameras)

History :: Camara Obscura

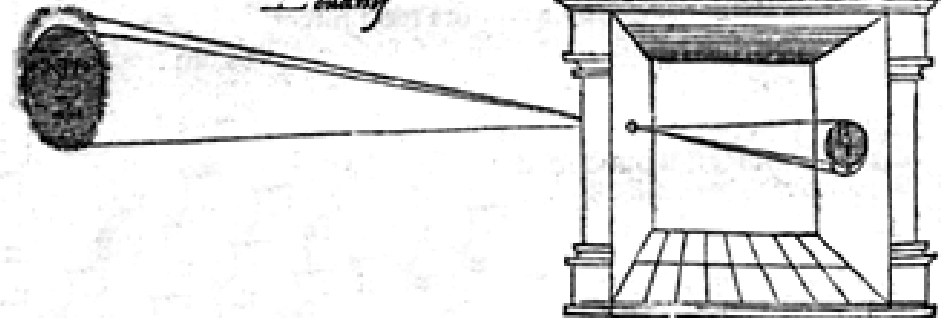
Camera obscura ([|Camera obscura) [LL. camera chamber + L. obscurus, obscura, dark.] (Opt.)

1. An apparatus in which the images of external objects, formed by a convex lens or a concave mirror, are thrown on a paper or other white surface placed in the focus of the lens or mirror within a darkened chamber, or box, so that the outlines may be traced.
2. (Photog.) An apparatus in which the image of an external object or objects is, by means of lenses, thrown upon a sensitized plate or surface placed at the back of an extensible darkened box or chamber variously modified; - commonly called simply the camera.

Websters Dictionary, 1913

illum in tabula per radios Solis, quam in cœlo contingit: hoc est, si in cœlo superior pars deliquiū patiatur, in radiis apparebit inferior deficere, vt ratio exigit optica.

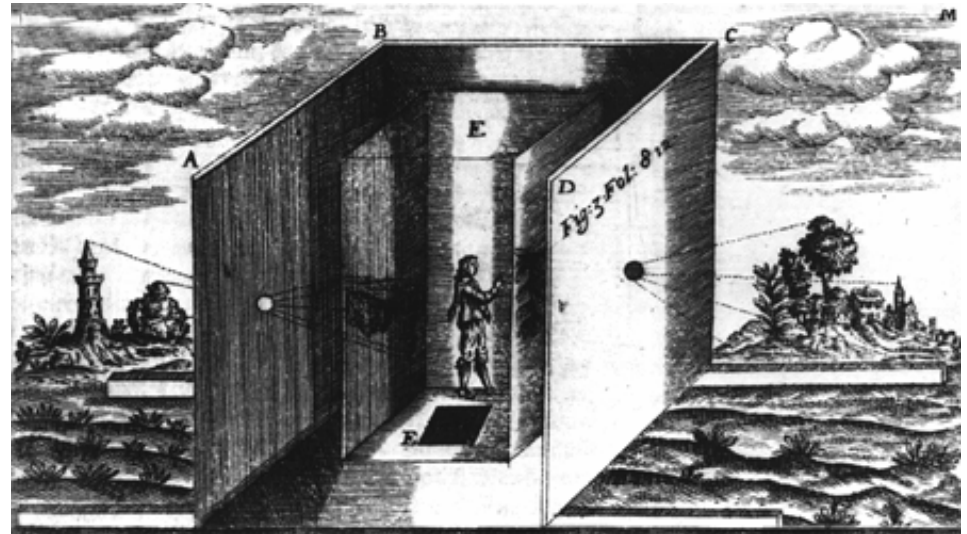
*Solis deliquium Anno Christi
1544. Die 24. Januarij
Louanij*



Sic nos exactè Anno .1544. Louanii eclipsim Solis obseruauimus, inuenimusq; deficere paulò plus q̄ dex-

Camera Obscura, Reinerus Gemma-Frisius, 1544, published in "De Radio Astronomica et Geometrica", 1545 from Gernsheim, H., "The Origins of Photography"

History :: Camera Obscura

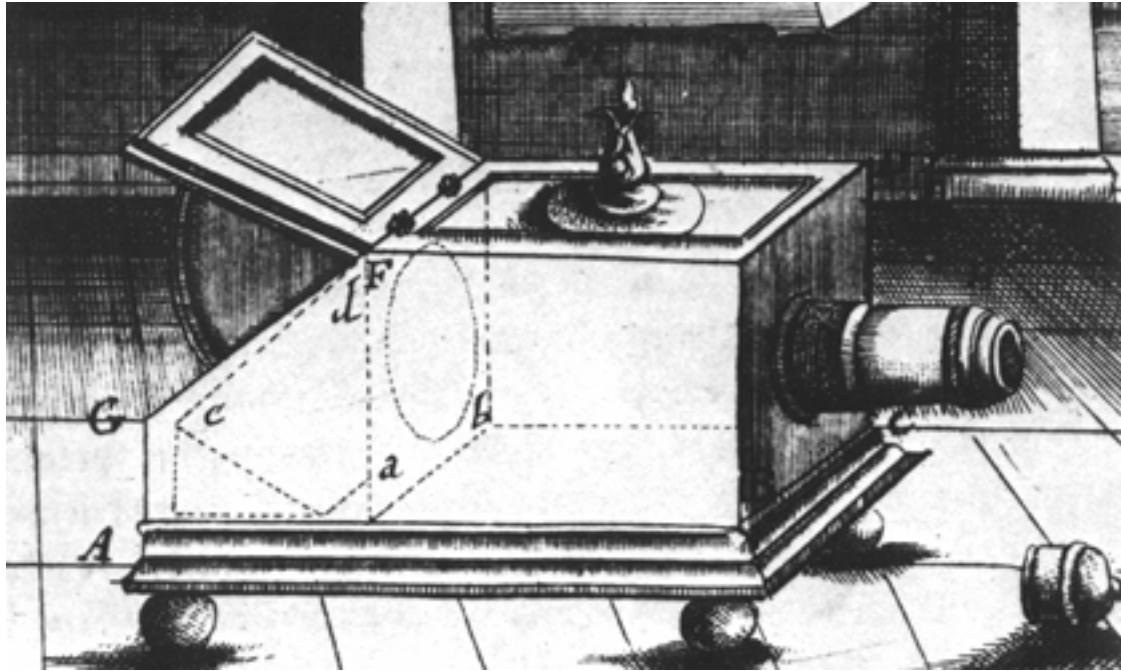


Camera Obscura, Athanasius Kircher, 1646
from Gernsheim, H., *"The Origins of Photography"*

Portable 'Tent' Camera Obscura, Johannes Kepler, 1620

from Gernsheim, H., *"The Origins of Photography"*

History :: Camera Obscura



Reflex Camera Obscura, Johannes Zahn, 1685

published in Zahn, J., "*Oculus Artificialis*", (1685 - 1686)

from Gernsheim, H., "*The Origins of Photography*"

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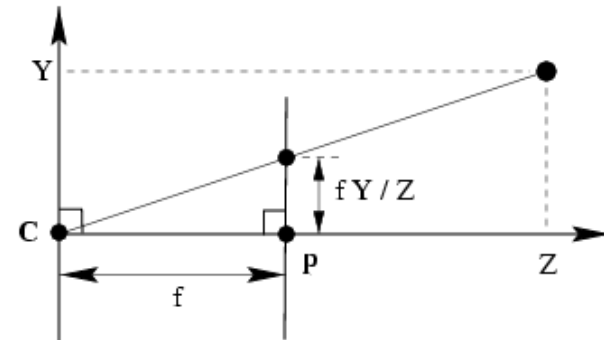
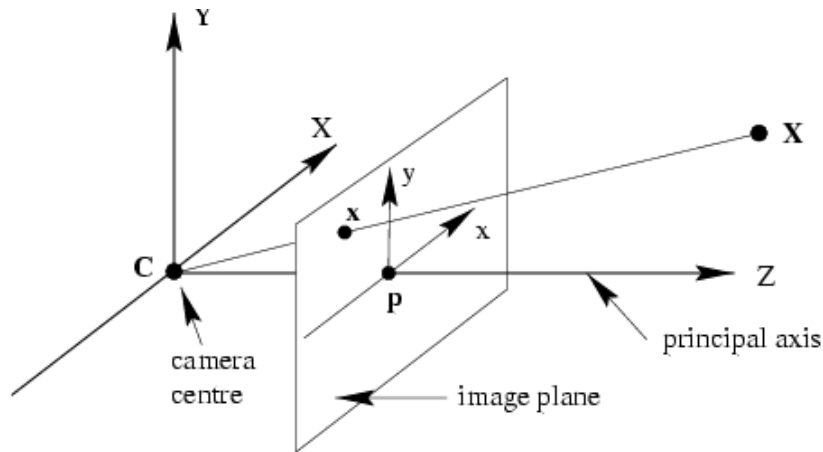
Intensity Image

- photometric parameters
 - light energy reaching the sensors after being reflected from objects (not discussed in this course)
- **geometric parameters**
 - 2D position in the image on which a 3D point is projected (depending on camera geometry, discussed here)

Outline – Camera Model

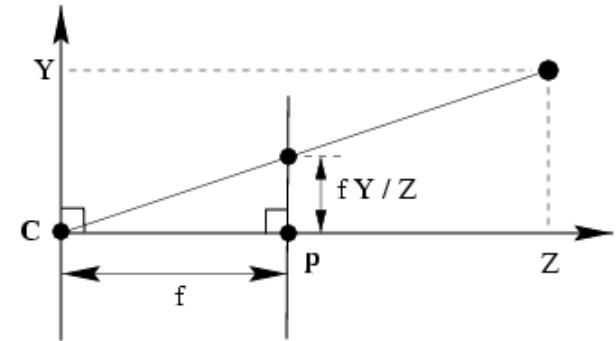
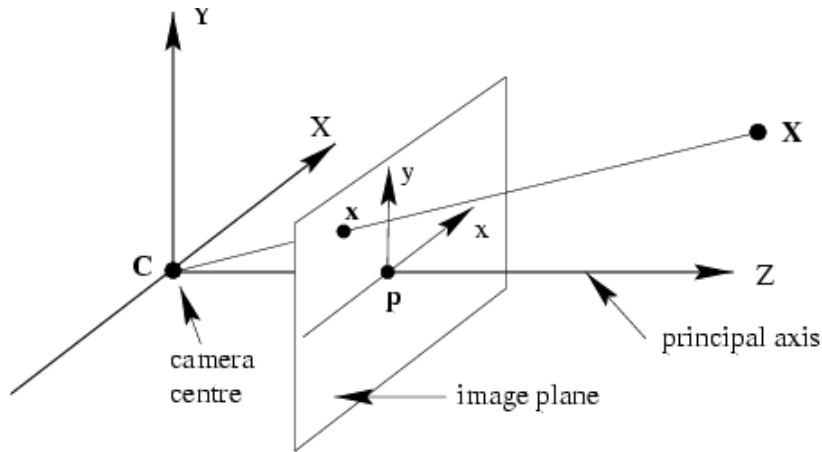
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Pinhole Camera Model



- mapping between 3D world and 2D image
 - central projection
 - models are described in matrices with particular properties
- $$(X, Y, Z)^T \mapsto (x, y)^T$$

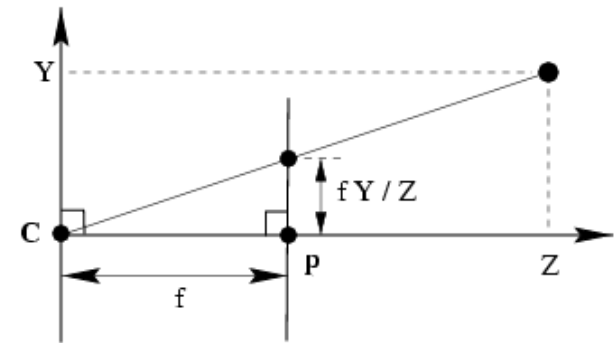
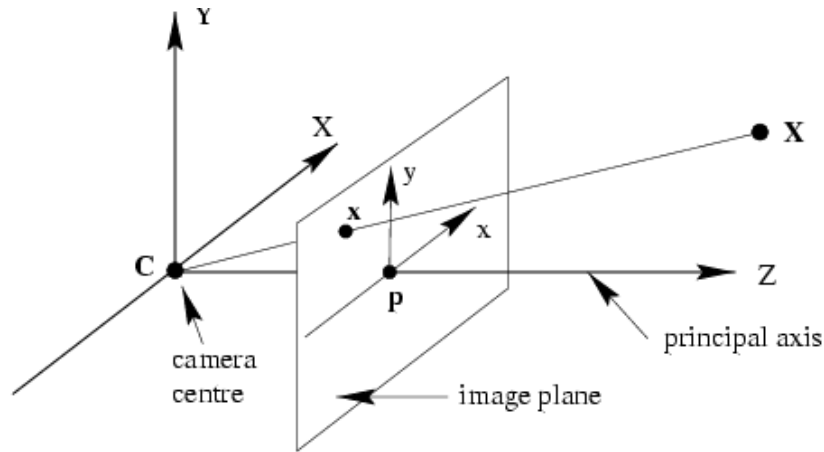
Homogeneous Coordinates



$$(X, Y, Z, 1)^T \mapsto (x, y, z)^T$$

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Central Projection

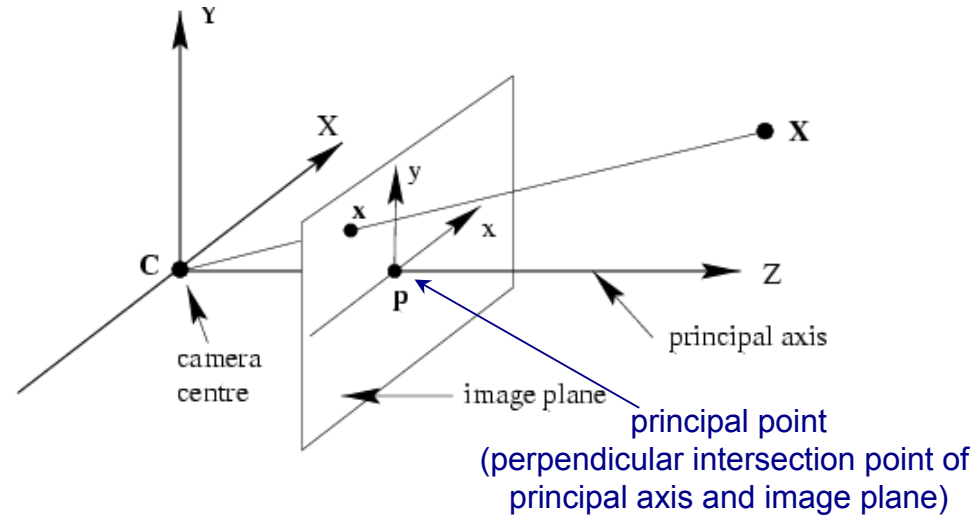
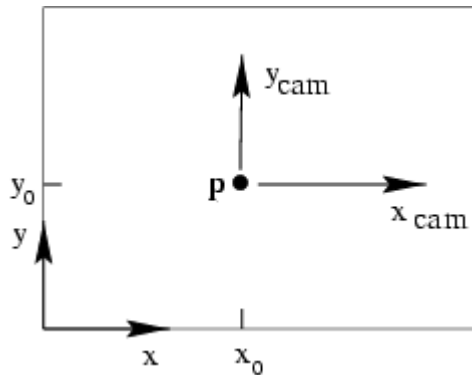


$$\begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & & \\ & f & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

$$\mathbf{P} = \text{diag}(f, f, 1)[\mathbf{I} \mid \mathbf{0}]$$

Principal Point Offset



$$(X, Y, Z)^T \mapsto (fX/Z + p_x, fY/Z + p_y)^T$$

where $(p_x, p_y)^T$

are the coordinates of the principal point

Principal Point Offset

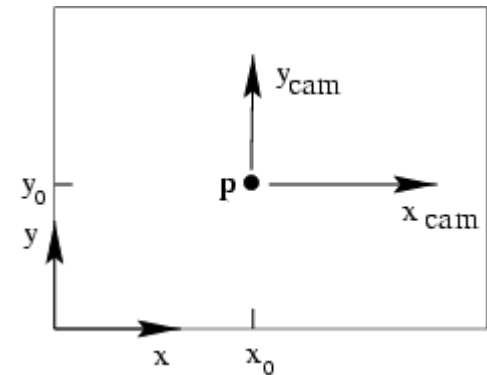
$$\begin{pmatrix} fX + Zp_x \\ fY + Zp_y \\ Z \end{pmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$\mathbf{x} = \mathbf{K}[\mathbf{I} \mid \mathbf{0}]\mathbf{X}$$

where

$$\mathbf{K} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix}$$

is called camera calibration matrix



Camera Rotation and Translation

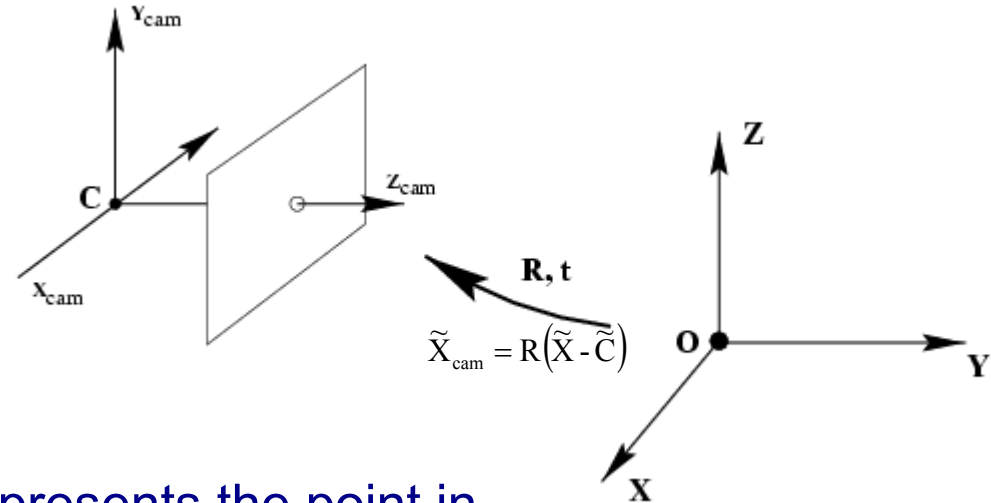
inhomogeneous coordinates

$$\tilde{\mathbf{X}}_{\text{cam}} = \mathbf{R}(\tilde{\mathbf{X}} - \tilde{\mathbf{C}})$$

where $\tilde{\mathbf{X}} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$

$$\tilde{\mathbf{X}}_{\text{cam}} = \begin{pmatrix} X_{\text{cam}} \\ y_{\text{cam}} \\ Z_{\text{cam}} \end{pmatrix}$$

$$\tilde{\mathbf{C}}$$



represents the point in world coordinates

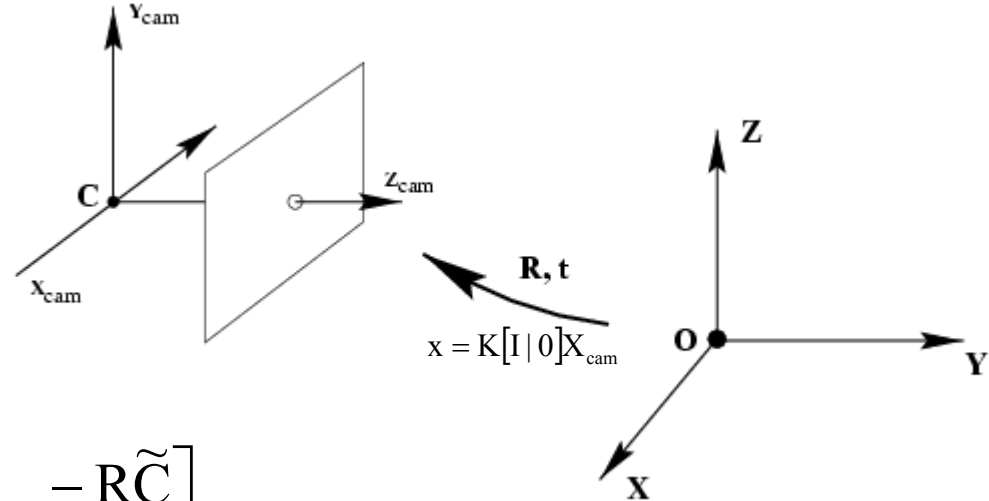
represents the same point in camera coordinates

represents the coordinates of the camera origin in the world coordinate frame

Camera Rotation and Translation

homogeneous coordinates

$$\mathbf{X}, \mathbf{X}_{cam} \in \mathbb{R}^4$$



$$\mathbf{X}_{cam} = \begin{bmatrix} \mathbf{R} & -\mathbf{R}\tilde{\mathbf{C}} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{bmatrix} \mathbf{R} & -\mathbf{R}\tilde{\mathbf{C}} \\ 0 & 1 \end{bmatrix} \mathbf{X}$$

$$\mathbf{x} = \mathbf{K}[\mathbf{I} \mid 0] \mathbf{X}_{cam} \quad \text{projection to image plane from camera coordinates}$$

$$\mathbf{x} = \mathbf{K}\mathbf{R}[\mathbf{I} \mid -\tilde{\mathbf{C}}] \mathbf{X} \quad \text{projection to image plane from world coordinates}$$

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

Extrinsic and Intrinsic Parameters

$$x = PX$$

where

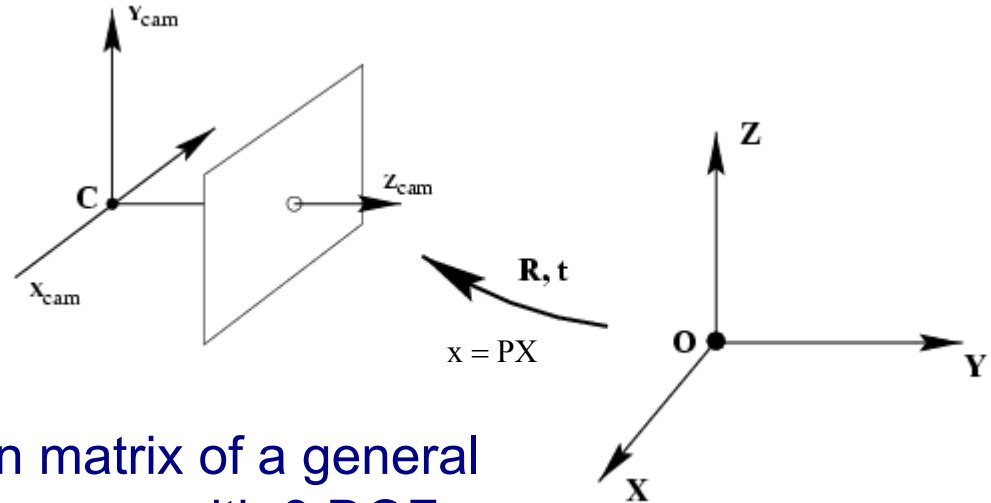
$$P = KR \left[I \mid -\tilde{C} \right]$$

projection matrix of a general pinhole camera with 9 DOF

$$K = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$$

intrinsic camera parameters with 3 DOF

R, \tilde{C} extrinsic camera parameters with each 3 DOF
(camera orientation and position in world coordinates)



Camera Rotation and Translation

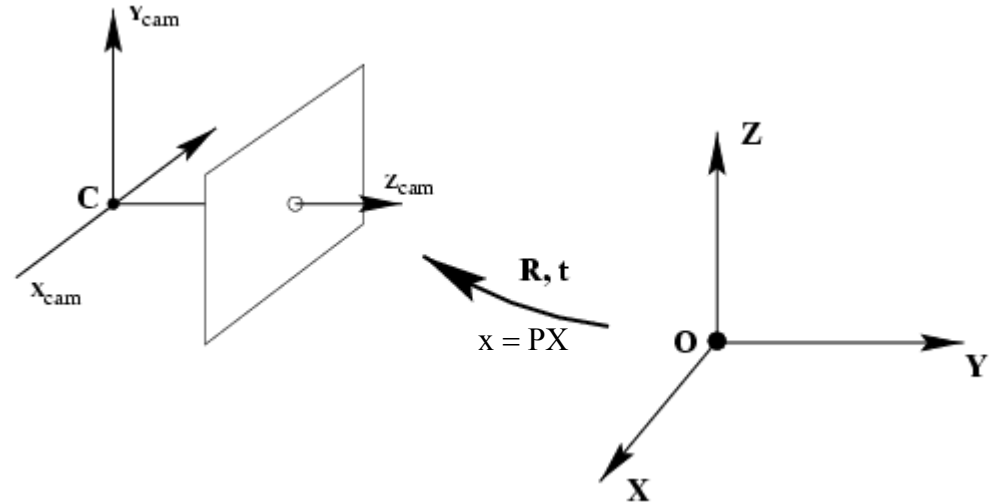
no explicit
camera center

$$\tilde{\mathbf{X}}_{cam} = R\tilde{\mathbf{X}} + t$$

$$\begin{aligned} \mathbf{P} &= \mathbf{K}[\mathbf{R} \mid \mathbf{t}] \\ &= \mathbf{K}[\mathbf{R} \mid -R\tilde{\mathbf{C}}] \end{aligned}$$

where

$$\mathbf{t} = -R\tilde{\mathbf{C}} \quad \text{from} \quad \mathbf{x} = \mathbf{KR}[\mathbf{I} \mid -\tilde{\mathbf{C}}]\mathbf{X}$$



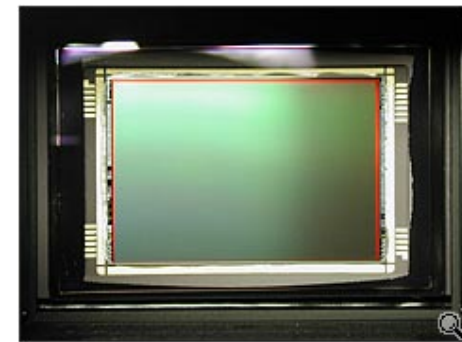
CCD Cameras :: Non-Square Pixel

m_x, m_y number of pixels per unit distance

$$K = \begin{bmatrix} m_x & & & \\ & m_y & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} f & p_x \\ f & p_y \\ & & & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} \alpha_x & & p_x \\ & \alpha_y & p_y \\ & & 1 \end{bmatrix} \quad 4 \text{ DOF}$$

$$P = KR \begin{bmatrix} I & | & -\tilde{C} \end{bmatrix} \quad 10 \text{ DOF}$$



Skew Parameter

S skew parameter

$$K = \begin{bmatrix} \alpha_x & s & p_x \\ & \alpha_y & p_y \\ & & 1 \end{bmatrix} \quad 5 \text{ DOF}$$

$$P = KR \begin{bmatrix} I & | & -\tilde{C} \end{bmatrix} \quad \text{finite projective camera with 11 DOF}$$

Finite Projective Camera :: Summary

$$\mathbf{x} = P\mathbf{X}$$

$$P = \begin{bmatrix} \alpha_x & s & p_x \\ & \alpha_y & p_y \\ & & 1 \end{bmatrix} \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix}$$

projection matrix
11 DOF (5+3+3)

$$= \underbrace{KR}_{\text{non-singular}} \begin{bmatrix} I & | & -\tilde{C} \end{bmatrix}$$

non-singular

Finite Projective Camera :: Decomposition of P

$$\mathbf{M} = \mathbf{KR}$$

non-singular 3x3 matrix (8 DOF)

$$\mathbf{P} = \mathbf{KR} \left[\mathbf{I} \mid -\tilde{\mathbf{C}} \right]$$

$$\mathbf{P} = \mathbf{M} \left[\mathbf{I} \mid -\tilde{\mathbf{C}} \right]$$

decompose projection matrix P in K,R,C

$$\mathbf{P} = \left[\mathbf{M} \mid \mathbf{p}_4 \right]$$

$$\left[\mathbf{K}, \mathbf{R} \right] = \underbrace{RQ}(\mathbf{M})$$

$$\tilde{\mathbf{C}} = -\mathbf{M}^{-1} \mathbf{p}_4$$

RQ matrix
decomposition

Finite Projective Camera :: Summary

$$\begin{aligned} P &= KR \left[I \mid -\tilde{C} \right] && \text{where} && P \in \mathbb{R}^{3 \times 4} \\ &= M \left[I \mid -\tilde{C} \right] \end{aligned}$$

- camera matrix P are identical with the set of homogeneous 3×4 matrix for which the left hand 3×3 submatrix is non singular
- $\{\text{finite cameras}\} = \{P \mid \det M \neq 0\} = \{P \mid \text{rank}(M) = 3\}$
- If $\text{rank}(P) = 3$, but $\text{rank}(M) < 3$, then camera at infinity
- if $\text{rank}(P) < 3$ the matrix mapping will be a line or a point and not a plane (not a 2D image)

Outline – Camera Model

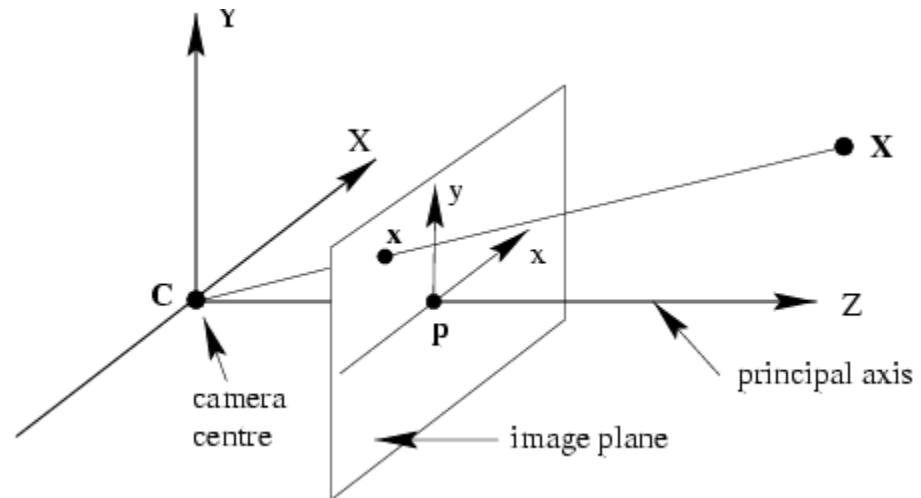
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Camera Anatomy

- camera center
- column vectors
- principal plane
- axis plane
- principal point
- principal ray



Camera Center

$$PC = 0 \quad P \text{ has a 1D null-space}$$

we will prove that the 4-vector C is the camera center

$$X(\lambda) = \lambda A + (1 - \lambda)C \quad \text{points on a line through } A \text{ and } C$$

$$x = PX = \lambda PA + (1 - \lambda)PC = \lambda PA \quad \text{since } PC = 0$$

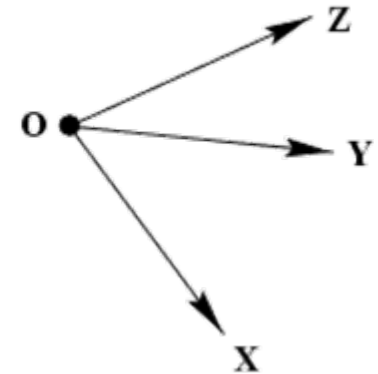
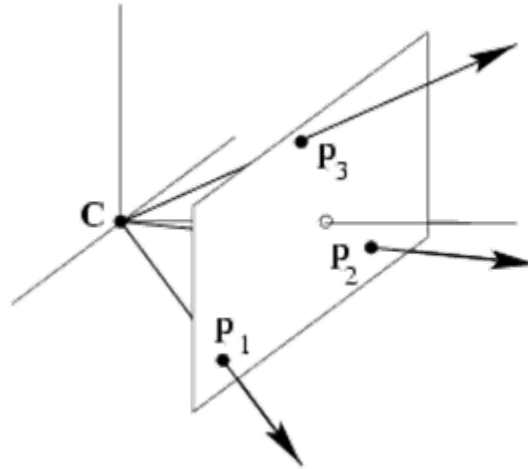
all 3D points on the line are mapped on the same 2D image point, and thus the line is a ray through the camera center

$$\text{Finite cameras:} \quad C = \begin{pmatrix} \tilde{C} \\ 1 \end{pmatrix} = \begin{pmatrix} -M^{-1}\mathbf{p}_4 \\ 1 \end{pmatrix}$$

$$\text{Infinite cameras:} \quad C = \begin{pmatrix} \mathbf{d} \\ 0 \end{pmatrix}, M\mathbf{d} = 0$$

Column Vectors

$$\mathbf{p}_i \in \mathbb{R}^3, i = 1, \dots, 4$$



column vectors are the image points, which project the axis directions (X,Y,Z) and the origin

example for the image of the y-axis (vanishing pt)

$$[\mathbf{p}_2] = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3 \quad \mathbf{p}_4] \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T$$

\mathbf{p}_4 is the image of the world origin

Row Vectors

$$\mathbf{p}^i \in \mathbb{R}^4, i = 1, \dots, 3$$

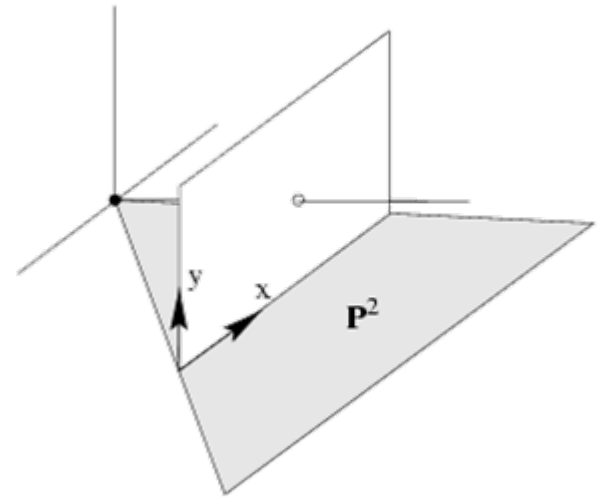
represent geometrically particular world planes

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} = \begin{bmatrix} \mathbf{p}^{1T} \\ \mathbf{p}^{2T} \\ \mathbf{p}^{3T} \end{bmatrix} \quad \text{row vectors}$$
$$= [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3 \quad \mathbf{p}_4] \quad \text{column vectors}$$

Row Vectors of the Projection Matrix

P^1 is defined by the camera center and the line $x=0$ on the image

P^2 is defined by the camera center and the line $y=0$ on the image



Example P^2

$$\begin{bmatrix} x \\ 0 \\ w \end{bmatrix} = \begin{bmatrix} \mathbf{p}^{1T} \\ \mathbf{p}^{2T} \\ \mathbf{p}^{3T} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

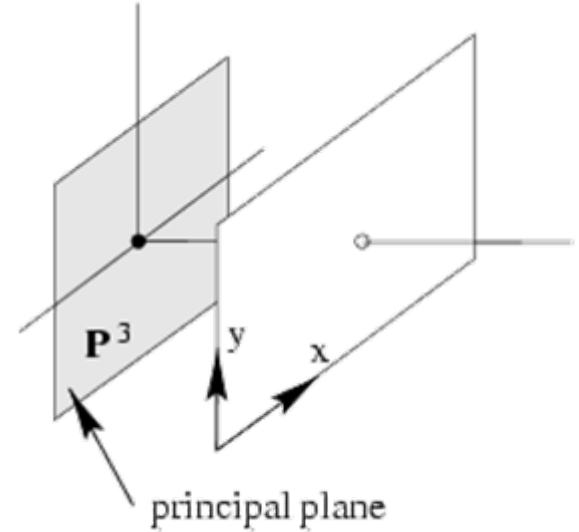
$$\mathbf{p}^{2T} \mathbf{X} = 0$$

$$\mathbf{p}^{2T} \mathbf{C} = 0$$

respectively for P^1

Principal Plane

plane through camera center and parallel to the image plane



$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{p}^{1T} \\ \mathbf{p}^{2T} \\ \mathbf{p}^{3T} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

points \mathbf{X} are imaged on the line at infinity

if \mathbf{X} is on the principle plane

$$\mathbf{p}^{3T} \mathbf{X} = 0$$

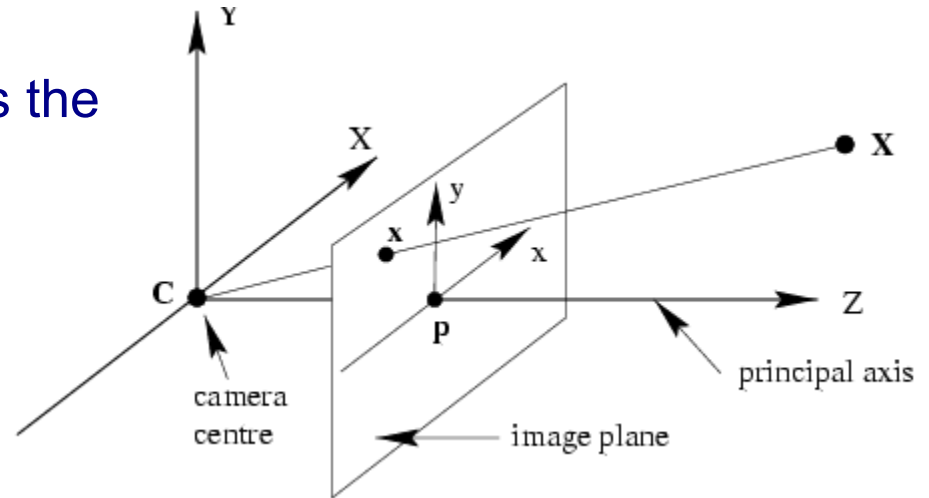
especially

$$\mathbf{p}^{3T} \mathbf{C} = 0$$

Principle Point

line through camera center and perpendicular to principle plane is the principle axis

intersection of the principle axis with the image plane is the principle point



$\hat{\mathbf{P}}^3 = (p_{31} \ p_{32} \ p_{33} \ 0)^T$ normal direction to principal plane

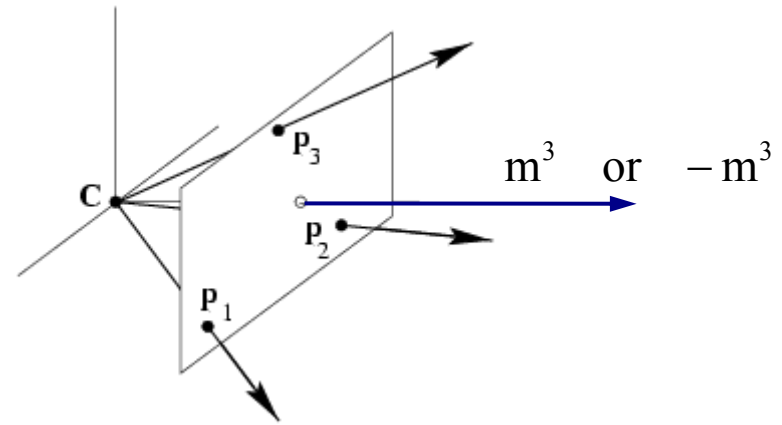
principle point

$$\mathbf{x}_0 = \mathbf{P}\hat{\mathbf{P}}^3 = \mathbf{M}\mathbf{m}^3 \quad \text{where} \quad \mathbf{P} = [\mathbf{M} \mid \mathbf{p}_4]$$

and \mathbf{m}^{3T} third row of \mathbf{M}

Principle Axis Vector

ambiguity that principle axis points towards the front of the camera (positive direction)



$$\mathbf{x} = \mathbf{P}_{\text{cam}} \mathbf{X}_{\text{cam}} = \mathbf{K} [\mathbf{I} \mid \mathbf{0}] \mathbf{X}_{\text{cam}}$$

$$\mathbf{v} = \det(\mathbf{K}) \mathbf{k}^3 = (0, 0, 1)^T$$

towards the front of the camera

$$\mathbf{P}_{\text{cam}} \mapsto k \mathbf{P}_{\text{cam}} \quad \mathbf{v} \mapsto k^4 \mathbf{v}$$

direction unaffected by scaling

$$\mathbf{P} = k \mathbf{K} \mathbf{R} [\mathbf{I} \mid -\tilde{\mathbf{C}}] = [\mathbf{M} \mid \mathbf{p}_4]$$

since $\det(\mathbf{R}) > 0$

$$\mathbf{v} = \det(\mathbf{M}) \mathbf{m}^3$$

Forward Projection

maps a point in space on the image plane $\mathbb{P}^3 \rightarrow \mathbb{P}^2$

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

Points at infinity: $\mathbf{D} = (\mathbf{d}^T, 0)^T$

$$\mathbf{x} = \mathbf{P}\mathbf{D} = [\mathbf{M} \mid \mathbf{p}_4] \mathbf{D} = \mathbf{M}\mathbf{d}$$

only \mathbf{M} effects the projection of points at infinity

Back-projection to Rays

points on the reconstructed ray

$$PC = 0 \quad \text{camera center } C$$

$$X = P^+ x \quad P^+ = P^T (PP^T)^{-1} \quad PP^+ = I$$

(pseudo-inverse)

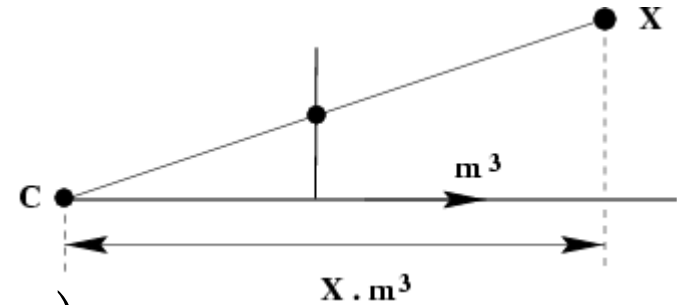
ray is the line formed by those two points

$$X(\lambda) = P^+ x + \lambda C$$

$$D = \left((M^{-1}x)^T, 0 \right)^T \quad \text{intersection of the ray with the plane at infinity}$$

$$X(\mu) = \underbrace{\mu \begin{pmatrix} M^{-1}x \\ 0 \end{pmatrix}}_D + \underbrace{\begin{pmatrix} -M^{-1}p_4 \\ 1 \end{pmatrix}}_C = \begin{pmatrix} M^{-1}(\mu x - p_4) \\ 1 \end{pmatrix}$$

Depth of Points



$$w = P^{3T} X = P^{3T} (X - C) = m^{3T} (\tilde{X} - \tilde{C})$$

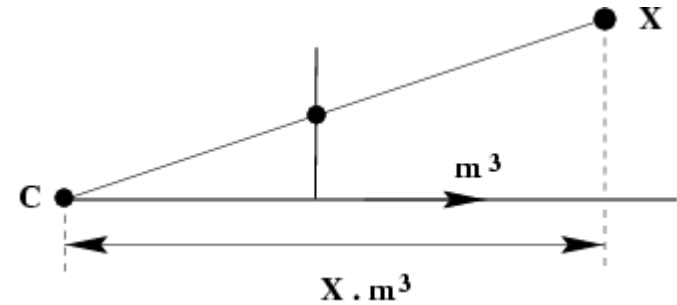
(PC=0) (dot product)

If $\det M > 0; \|m^3\| = 1$,
 then m^3 unit vector pointing in positive axis direction

Suppose $P(X, Y, Z, T)^T = w(x, y, 1)^T$. Then

$$\text{depth}(X; P) = \frac{\text{sign}(\det M) w}{T \|m^3\|}$$

Depth of Points: examples



$$\text{depth}(X; P) = \frac{\text{sign}(\det M)w}{\|m^3\|}$$

$$P(X, Y, Z, 1)^T = w(x, y, 1)^T$$

$$\mathbf{x} = P\mathbf{X}$$

$$\mathbf{x} = P\mathbf{X}$$

$$P = [I \mid 0]$$

$$P = R[I \mid -\tilde{C}]$$

$$\text{depth}(X; P) = Z$$

$$\text{depth}(X; P) = R^3{}^T (\tilde{X} - \tilde{C})$$

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Camera Matrix Decomposition

Finding the camera center C

$$PC = 0$$

numerically: find right null-space by SVD of P

$$P^{3 \times 4} = U^{3 \times 4} D^{4 \times 4} V^{4 \times 4 T} \rightarrow P^{3 \times 4} V^{4 \times 4} = U^{3 \times 4} D^{4 \times 4} \rightarrow PV^4 = 0$$

Algebraically:

$$X = \det([p_2, p_3, p_4]) \quad Y = -\det([p_1, p_3, p_4])$$

$$Z = \det([p_1, p_2, p_4]) \quad T = -\det([p_1, p_2, p_3])$$

where $C = (X, Y, Z, T)^T$

Camera Matrix Decomposition

Finding the camera center C

$$PC = 0$$

Any plane π going through C will be a linear combination of the three planes defined by the rows of P . Therefore:

$$\begin{aligned} \det([\pi, P^T]) &= 0 \\ &= \pi_1 \det(P_{234}) - \pi_2 \det(P_{134}) + \pi_3 \det(P_{124}) - \pi_4 \det(P_{123}) \end{aligned}$$

$$X = \det([p_2, p_3, p_4]) \quad Y = -\det([p_1, p_3, p_4])$$

$$Z = \det([p_1, p_2, p_4]) \quad T = -\det([p_1, p_2, p_3])$$

where $C = (X, Y, Z, T)^T$

Camera Matrix Decomposition

Finding the camera orientation and internal parameters

$$P = [M \mid -M\tilde{C}] = K[R \mid -R\tilde{C}] \quad K = \begin{bmatrix} \alpha_x & s & y_0 \\ 0 & \alpha_y & x_0 \\ 0 & 0 & 1 \end{bmatrix}$$

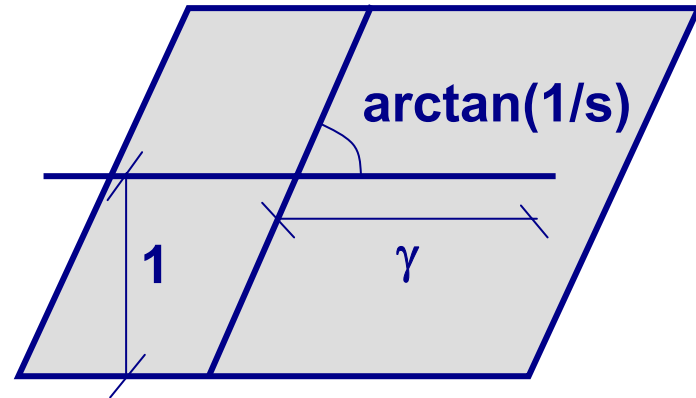
Decompose $M = KR$ using RQ decomposition

$$K = (Q \begin{matrix} \diagdown \\ R \\ \diagup \end{matrix})^{-1} = \begin{matrix} \diagdown \\ R^{-1} \\ \diagup \end{matrix} Q^{-1}$$

Ambiguity removed by enforcing positive diagonal entries

When is Skew Non-zero?

$$K = \begin{bmatrix} \alpha_x & s & p_x \\ & \alpha_y & p_y \\ & & 1 \end{bmatrix}$$



for CCD/CMOS, always $s=0$

Image from image, $s \neq 0$ possible
(non coinciding principal axis)

resulting camera: **HP**

where H is a 3x3 homography

Euclidean vs. Projective Spaces

general projective interpretation

$$P = \left[\begin{array}{c} 3 \times 3 \text{ homography} \\ 4 \times 4 \text{ homography} \end{array} \right] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Meaningful decomposition in K, R, t requires
Euclidean image and space

Camera center is still valid in projective space

Principal plane requires affine image and space

Principal ray requires affine image and Euclidean space

Outline – Camera Model

- History of Camera (Camera Obscura)
- Intensity Images
- Geometric Parameters of a Finite camera
- Projective Camera
 - Camera Center, Principal Plane, Axis Plane, Principal Point, Principal Ray
 - Decomposition of Camera Matrix
- Cameras at Infinity
- Other Camera Models (Pushbroom and Line Cameras)

Cameras at Infinity

Cameras with its center lying at infinity

$$P \begin{bmatrix} d \\ 0 \end{bmatrix} = 0 \Rightarrow \det M = 0 \quad M \text{ is singular}$$

Two types of cameras at infinity:
Affine and non-affine cameras

Affine Cameras

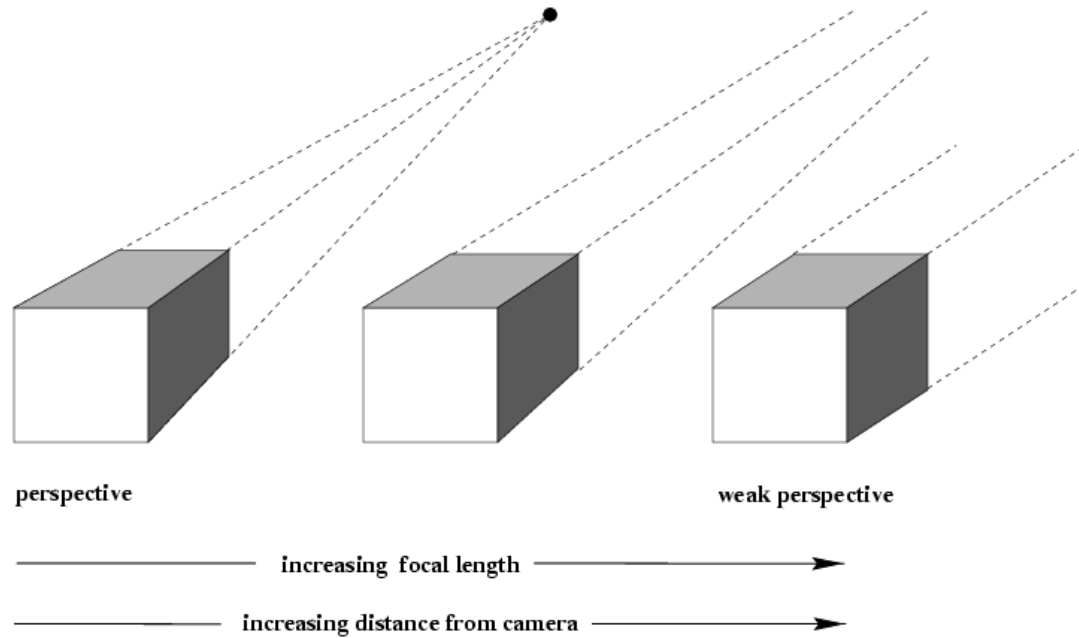
Definition:

An affine camera is a camera with a camera matrix P in which the last row p^{3T} is of the form $(0,0,0,1)^T$.

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

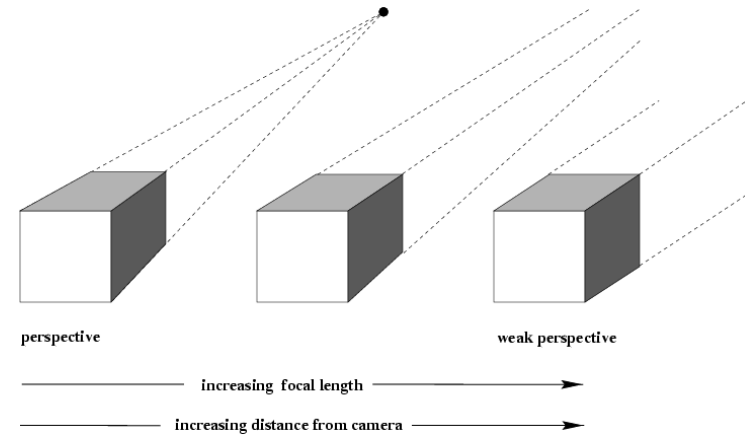
points at infinity are mapped to points at infinity

Affine Cameras



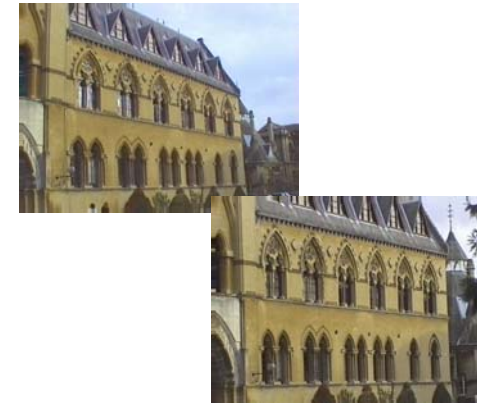
Affine Cameras

$$P_0 = KR \begin{bmatrix} I & -\tilde{C} \end{bmatrix} = K \begin{bmatrix} \mathbf{r}^{1T} & -\mathbf{r}^{1T} \tilde{C} \\ \mathbf{r}^{2T} & -\mathbf{r}^{2T} \tilde{C} \\ \mathbf{r}^{3T} & -\mathbf{r}^{3T} \tilde{C} \end{bmatrix}$$



$d_0 = -\mathbf{r}^{3T} \tilde{C}$ distance of the world origin from the camera center in direction of the principle ray

$$P_t = K \begin{bmatrix} \mathbf{r}^{1T} & -\mathbf{r}^{1T} (\tilde{C} - t\mathbf{r}^3) \\ \mathbf{r}^{2T} & -\mathbf{r}^{2T} (\tilde{C} - t\mathbf{r}^3) \\ \mathbf{r}^{3T} & -\mathbf{r}^{3T} (\tilde{C} - t\mathbf{r}^3) \end{bmatrix} = K \begin{bmatrix} \mathbf{r}^{1T} & -\mathbf{r}^{1T} \tilde{C} \\ \mathbf{r}^{2T} & -\mathbf{r}^{2T} \tilde{C} \\ \mathbf{r}^{3T} & d_t \end{bmatrix}$$



modifying p_{34} corresponds to moving along principal ray

Affine Cameras

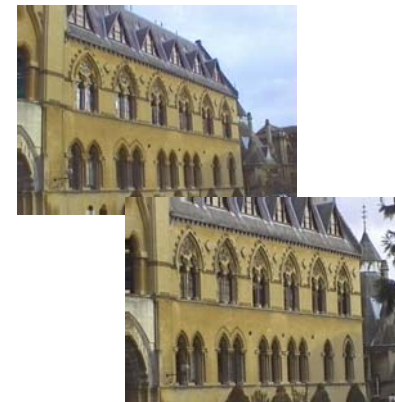
combine tracking back and zooming

$$P_t = K \begin{bmatrix} d_t / d_0 & & & \\ & d_t / d_0 & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r}^{1T} & -\mathbf{r}^{1T} \tilde{\mathbf{C}} \\ \mathbf{r}^{2T} & -\mathbf{r}^{2T} \tilde{\mathbf{C}} \\ \mathbf{r}^{3T} & d_t \end{bmatrix}$$

magnification factor $k=d_t/d_0$
remains fixed

$$= \frac{d_t}{d_0} K \begin{bmatrix} \mathbf{r}^{1T} & -\mathbf{r}^{1T} \tilde{\mathbf{C}} \\ \mathbf{r}^{2T} & -\mathbf{r}^{2T} \tilde{\mathbf{C}} \\ \mathbf{r}^{3T} d_0 / d_t & d_0 \end{bmatrix}$$

$$P_\infty = \lim_{t \rightarrow \infty} P_t = K \begin{bmatrix} \mathbf{r}^{1T} & -\mathbf{r}^{1T} \tilde{\mathbf{C}} \\ \mathbf{r}^{2T} & -\mathbf{r}^{2T} \tilde{\mathbf{C}} \\ \mathbf{0} & d_0 \end{bmatrix}$$



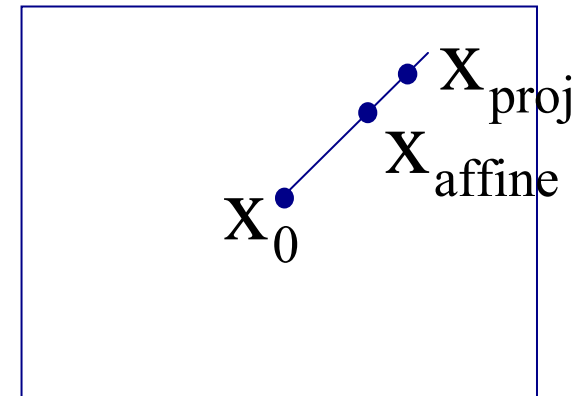
Error in Employing Affine Cameras

$$\mathbf{X} = \begin{pmatrix} \alpha r^1 + \beta r^2 \\ 1 \end{pmatrix}$$
 point on plane parallel with principal plane and through origin, then $\mathbf{P}_0 \mathbf{X} = \mathbf{P}_t \mathbf{X} = \mathbf{P}_\infty \mathbf{X}$

$$\mathbf{X} = \begin{pmatrix} \alpha r^1 + \beta r^2 + \Delta r^3 \\ 1 \end{pmatrix}$$
 general points (not on the parallel plane) with distance Δ from the plane

$$\mathbf{x}_{\text{proj}} = \mathbf{P}_0 \mathbf{X} = \mathbf{K} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ d_0 + \Delta \end{pmatrix}$$

$$\mathbf{x}_{\text{affine}} = \mathbf{P}_\infty \mathbf{X} = \mathbf{K} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ d_0 \end{pmatrix}$$



Error in Employing Affine Cameras

$$K = \begin{bmatrix} K_{2 \times 2} & \tilde{x}_0 \\ \tilde{0}^T & 1 \end{bmatrix}$$

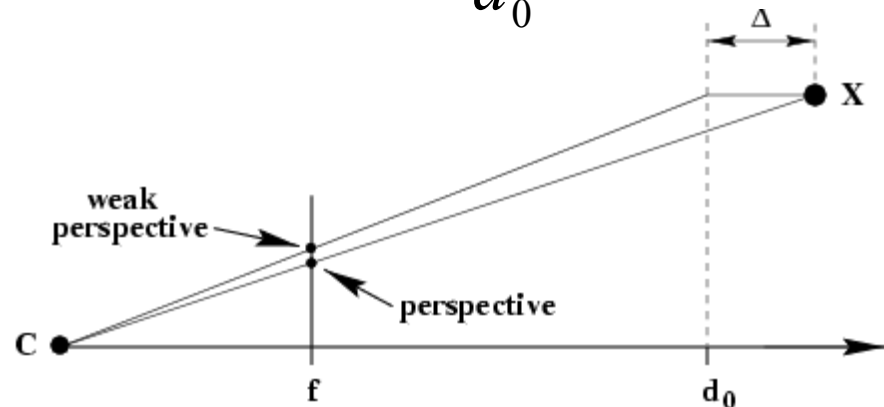
$$X_{\text{proj}} = \begin{pmatrix} K_{2 \times 2} \tilde{X} + (d_0 + \Delta) \tilde{X}_0 \\ d_0 + \Delta \end{pmatrix}$$

$$X_{\text{affine}} = \begin{pmatrix} K_{2 \times 2} \tilde{X} + d_0 \tilde{X}_0 \\ d_0 \end{pmatrix}$$

$$\tilde{X}_{\text{proj}} = \tilde{X}_0 + \frac{K_{2 \times 2} \tilde{X}}{d_0 + \Delta}$$

$$\tilde{X}_{\text{affine}} = \tilde{X}_0 + \frac{K_{2 \times 2} \tilde{X}}{d_0}$$

$$\tilde{X}_{\text{affine}} - \tilde{X}_0 = \frac{d_0 + \Delta}{d_0} (\tilde{X}_{\text{proj}} - \tilde{X}_0)$$



Affine Imaging Conditions

$$\mathbf{X}_{\text{affine}} - \mathbf{X}_{\text{proj}} = \frac{\Delta}{d_0} (\mathbf{X}_{\text{proj}} - \mathbf{X}_0)$$

approximation should only cause small error

1. Δ much smaller than d_0
2. points close to principal ray (i.e. small field of view)

Decomposition of P_∞

$$P_\infty = \begin{bmatrix} K_{2 \times 2} & \tilde{x}_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & \tilde{t} \\ 0 & d_0 \end{bmatrix} = \begin{bmatrix} d_0^{-1} K_{2 \times 2} & \tilde{x}_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & \tilde{t} \\ 0 & 1 \end{bmatrix}$$

absorb d_0 in $K_{2 \times 2}$

$$= \begin{bmatrix} K_{2 \times 2} \tilde{R} & K_{2 \times 2} \tilde{t} + \tilde{x}_0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} K_{2 \times 2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & \tilde{t} + K_{2 \times 2}^{-1} \tilde{x}_0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} K_{2 \times 2} & K_{2 \times 2} \tilde{t} + \tilde{x}_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & 0 \\ 0 & 1 \end{bmatrix}$$

$$P_\infty = \begin{bmatrix} K_{2 \times 2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & \tilde{t} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} K_{2 \times 2} & \tilde{x}_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & 0 \\ 0 & 1 \end{bmatrix}$$

alternatives, because 8dof (3+3+2), not more

Summary of Parallel Projections

$$\mathbf{P}_{\infty} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

canonical representation

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{2 \times 2} & 0 \\ 0 & 1 \end{bmatrix}$$

calibration matrix

principal point is not defined

Hierarchy of Affine Cameras

$$\mathbf{P}_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{dropping the z-coordinate}$$

orthographic projection

$$\mathbf{H} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix}$$
$$\mathbf{P}_\infty = \begin{bmatrix} \mathbf{r}^{1T} & t_1 \\ \mathbf{r}^{1T} & t_2 \\ 0 & 1 \end{bmatrix} \quad (5\text{dof})$$

Hierarchy of Affine Cameras

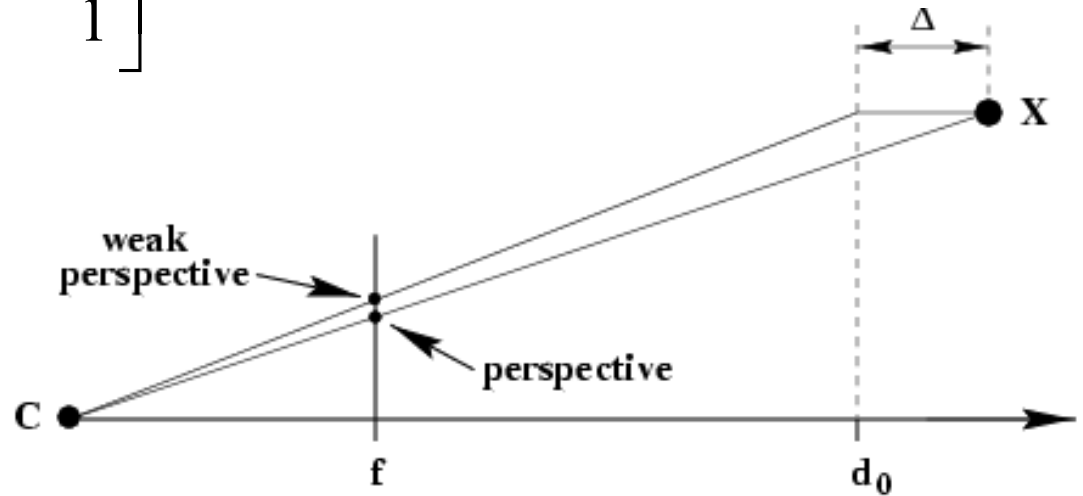
scaled orthographic projection

$$\mathbf{P}_\infty = \begin{bmatrix} k & & & \\ & k & & \\ & & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r}^{1T} & t_1 \\ \mathbf{r}^{1T} & t_2 \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{r}^{1T} & t_1 \\ \mathbf{r}^{1T} & t_2 \\ \mathbf{0} & 1/k \end{bmatrix} \quad (6\text{dof})$$

Hierarchy of Affine Cameras

weak perspective projection

$$P_{\infty} = \begin{bmatrix} \alpha_x & & & \\ & \alpha_y & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} r^{1T} & t_1 \\ r^{2T} & t_2 \\ 0 & 1 \end{bmatrix} \quad (7\text{dof})$$



Hierarchy of Affine Cameras

affine camera

$$P_A = \begin{bmatrix} \alpha_x & s & & \\ & \alpha_y & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} r^{1T} & t_1 \\ r^{2T} & t_2 \\ 0 & 1 \end{bmatrix} \quad (8\text{dof})$$

$$P_A = \begin{bmatrix} m_{11} & m_{12} & m_{13} & t_1 \\ m_{21} & m_{22} & m_{23} & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

full generality of an affine camera

$$P_A = [3 \times 3 \text{ affine}] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} [4 \times 4 \text{ affine}]$$

- Affine camera is a projective camera with principal plane at infinity
- Affine camera maps parallel world lines to parallel image lines
- No center of projection, but direction of projection $P_A D=0$

General Camera at Infinity

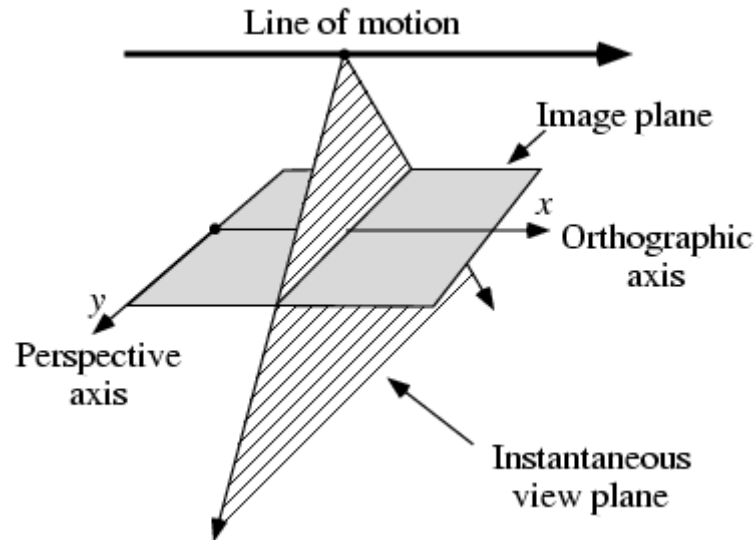
$$\mathbf{P} = [\mathbf{M} \mid \mathbf{p}_4] \quad \mathbf{M} \text{ is singular, but last row not zero}$$

- camera center is on plane at infinity
- principal plane is not plane at infinity
- images of points at infinity are in general not mapped to infinity on the image plane

Outline – Camera Model

- History of Camera (Camera Obscura)
- Intensity Images
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Pushbroom Cameras



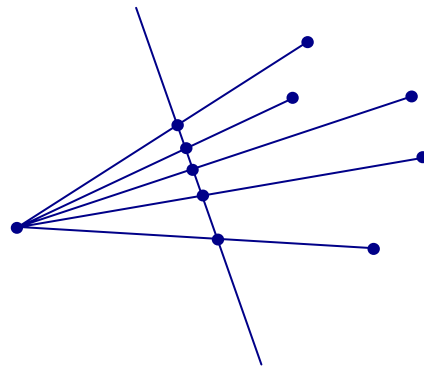
(11dof)

$$X = (X, Y, X, T)^T \quad PX = (x, y, w)^T \quad (x, y/w)^T$$

Straight lines are not mapped to straight lines!
(otherwise it would be a projective camera)

Line Cameras

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad (5\text{dof})$$



Null-space $PC=0$ yields camera center

Also decomposition $P_{2 \times 3} = K_{2 \times 2} R_{2 \times 2} [I_{2 \times 2} \mid -\tilde{c}]$

Summary Camera Models

- photometric and radiometric properties of a camera
- geometric parameters of a finite camera
- projective cameras
 - camera anatomy (camera center, principle plane, principle point, and principle axis)
 - camera matrix decomposition (camera center, orientation, and intrinsic parameter)
- cameras at infinity
 - affine cameras
 - non – affine cameras
- alternative models (pushbroom cameras, line cameras)

Literature on Camera Models

- Chapter 6 in R. Hartley and A. Zisserman, “*Multiple View Geometry*”, 2nd edition, Cambridge University Press, 2003.
- Chapter 3 in O. Faugeras, “*Three-dimensional Computer Vision*”, MIT Press, 1993.
- Chapter 2 in E. Trucco and A. Verri, “*Introductory Techniques for 3-D Computer Vision*”, Prentice Hall, 1998.
- H. Gernsheim, “*The Origins of Photography*”, *Thames and Hudson*, 1982.
- A. Shashua. “*Geometry and Photometry in 3D Visual Recognition*”, Ph.D. Thesis, MIT, Nov. 1992. AITR-1401.

Action of Projective Camera on Points

Forward Projection

maps a point in space on the image plane $\mathbb{P}^3 \rightarrow \mathbb{P}^2$

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

vanishing points $\mathbf{D} = (\mathbf{d}^T, 0)^T$

$$\mathbf{x} = \mathbf{P}\mathbf{D} = [\mathbf{M} \mid \mathbf{p}_4]\mathbf{D} = \mathbf{M}\mathbf{d}$$

only \mathbf{M} effects the projection of vanishing points

Action of Projective Camera on Points

Back-projection to Rays

$$\mathbf{X} = \mathbf{P}^+ \mathbf{x} \quad \mathbf{P}^+ = \mathbf{P}^T (\mathbf{P}\mathbf{P}^T)^{-1} \quad \mathbf{P}\mathbf{P}^+ = \mathbf{I}$$

(pseudo-inverse)

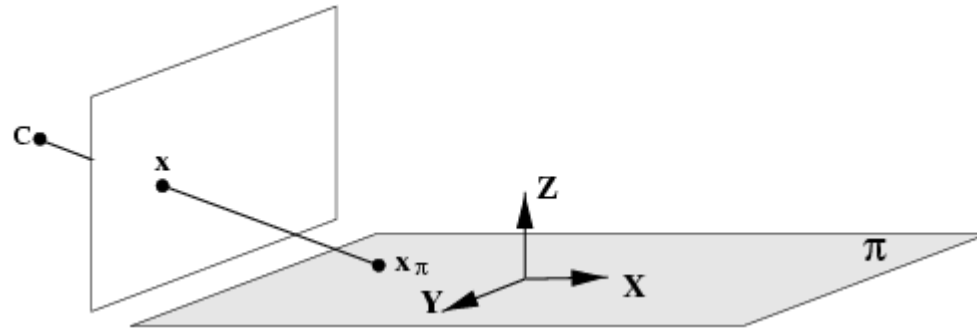
$$\mathbf{P}\mathbf{C} = \mathbf{0} \quad \text{camera center } \mathbf{C}$$

ray is the line formed by those two points

points on the reconstructed ray

$$\mathbf{X}(\lambda) = \mathbf{P}^+ \mathbf{x} + \lambda \mathbf{C}$$

Action of Projective Camera on Planes



$$x = PX = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_4 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

The most general transformation that can occur between a scene plane and an image plane under perspective imaging is a plane projective transformation

(affine camera-affine transformation)

Action of Projective Camera on Lines

forward projection

$$X(\mu) = P(A + \mu B) = PA + \mu PB = a + \mu b$$

back-projection

$$\Pi = P^T \mathbf{1}$$

$$\Pi^T X = \mathbf{1}^T P X = 0$$

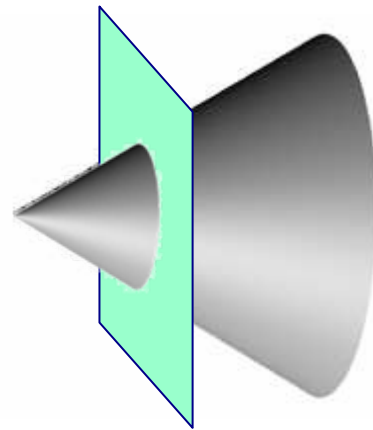
Action of Projective Camera on Conics

back-projection to cone

$$Q_{co} = P^T C P \quad X^T C X = X^T P^T C P X = 0$$

Note: vertex of degenerate quadric is optical center, since

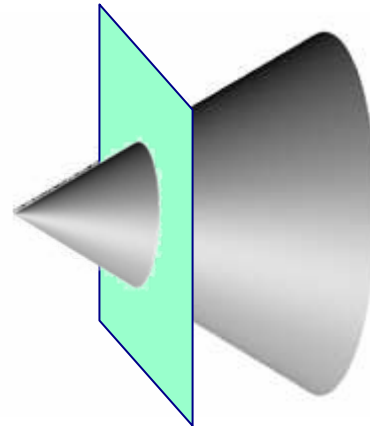
$$Q_{co} C = P^T C (P C) = 0$$



Action of Projective Camera on Conics

back-projection to cone

$$Q_{co} = P^T C P$$



Example $P = K[I | 0]$:

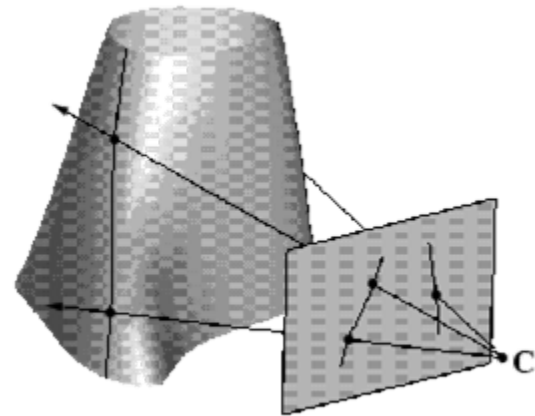
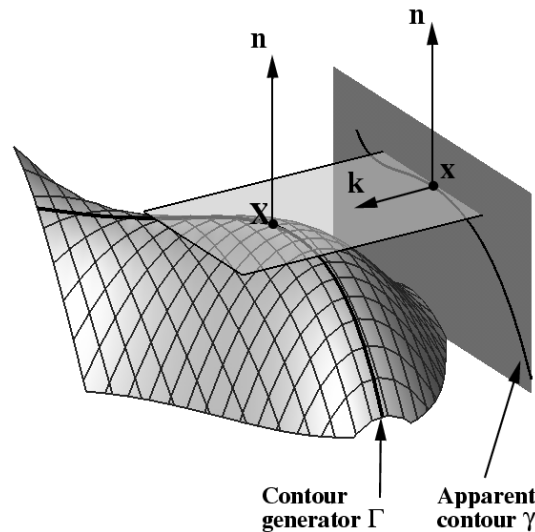
$$Q_{co} = \begin{bmatrix} K^T \\ 0 \end{bmatrix} C^T [K | 0] = \begin{bmatrix} K^T C K & 0 \\ 0 & 0 \end{bmatrix}$$

where $C = (0,0,0,1)^T$

Images of Smooth Surfaces

The contour generator Γ is the set of points X on S at which rays are tangent to the surface. The corresponding apparent contour γ is the set of points x which are the image of X , i.e. γ is the image of Γ

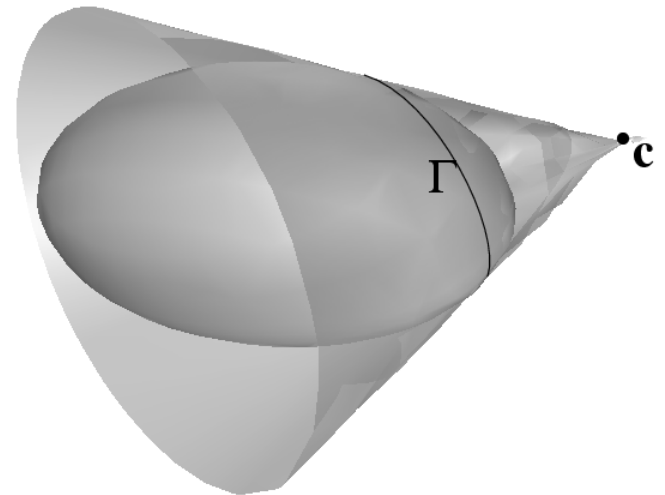
The contour generator Γ depends only on position of projection center, γ depends also on rest of P



Action of projective camera on quadrics

- Forward projection of quadrics:

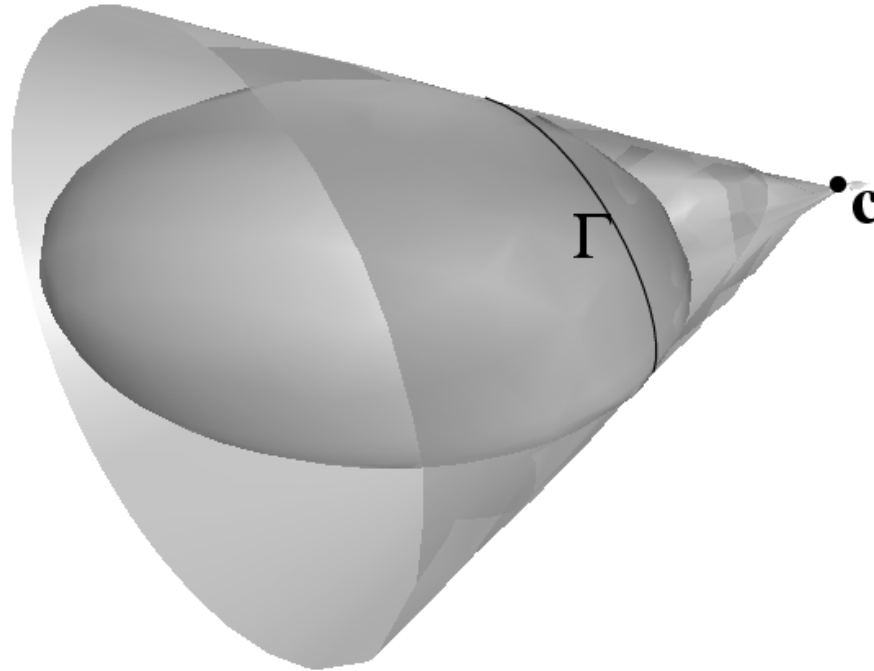
$$C^* = PQ^*P^T$$



- The plane of Γ for a quadric Q and camera center C is given by $P=QC$ (follows from pole-polar relation)

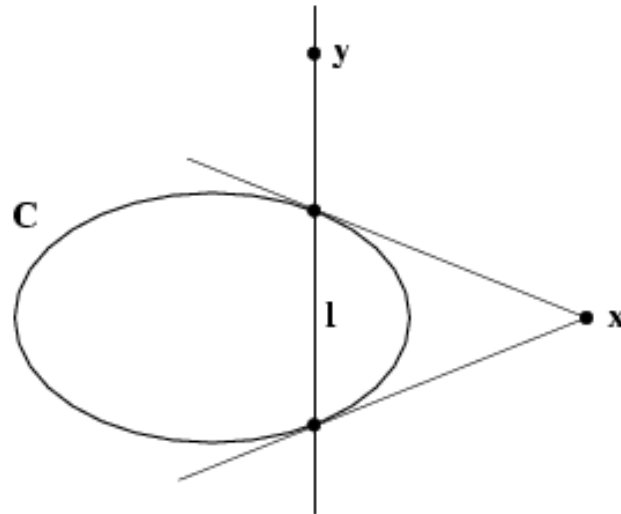
Pole-Polar Relationship for Planes and points with respect to quadrics

The polar plane $\Pi_{\Gamma} = QC$ of the point C with respect to the quadric Q intersects the quadric in a conic Γ . The lines tangent to Q at these points intersect at C .



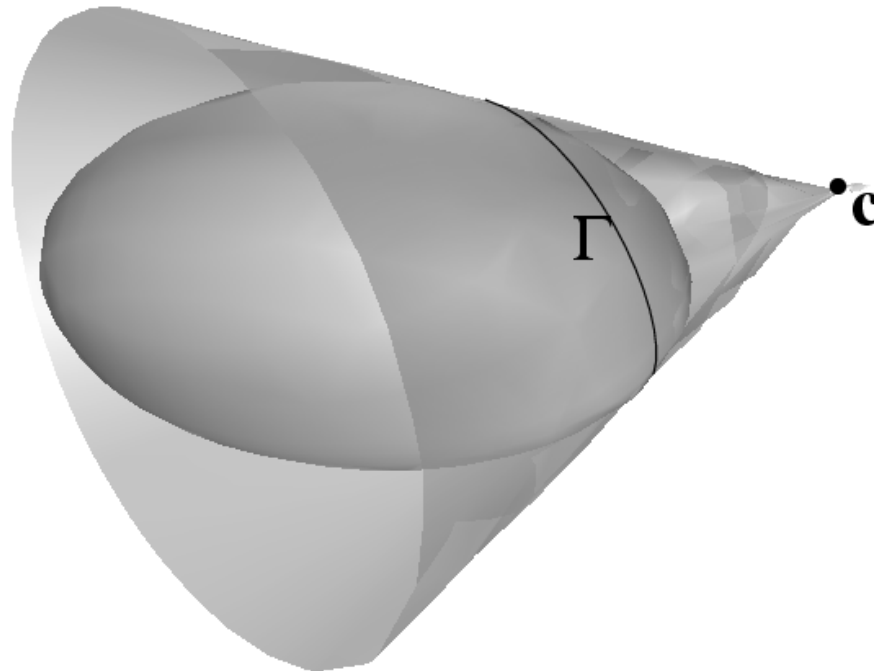
Reminder: Pole-Polar Relationship for lines and points with respect to conics

The polar line $l = Cx$ of the point x with respect to the conic C intersects the conic in two points. The two lines tangent to C at these points intersect at x



Pole-Polar Relationship for Planes and points with respect to quadrics

The polar plane $\Pi_{\Gamma} = \mathbf{QC}$ of the point C with respect to the quadric \mathbf{Q} intersects the quadric in a conic Γ . The planes tangent to \mathbf{Q} at the points on this conic intersect at C .

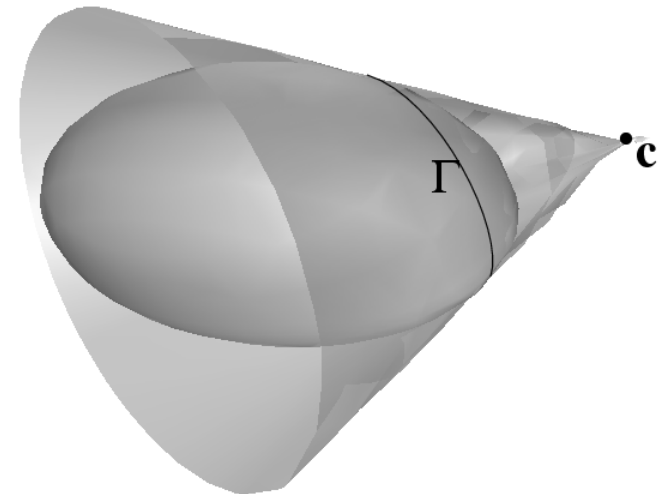


Action of Projective Camera on Quadrics

Forward Projection of Quadrics

$$C^* = PQ^*P^T$$

$$\Pi^T Q^* \Pi = \mathbf{1}^T P Q^* P^T \mathbf{1} = \mathbf{1}^T C^* \mathbf{1} = 0$$

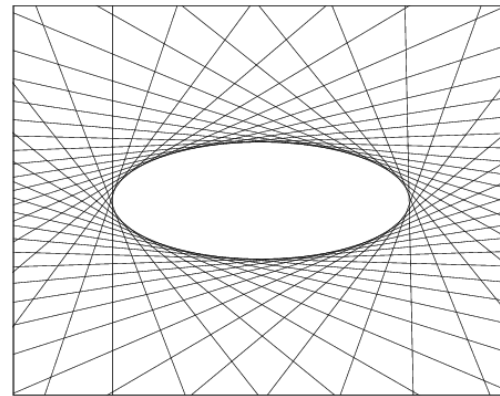


Action of projective camera on quadrics

- Forward projection of quadrics:

lines tangent to the conic \mathbf{C} satisfy:

$$\mathbf{1}^T \mathbf{C}^* \mathbf{1} = 0$$



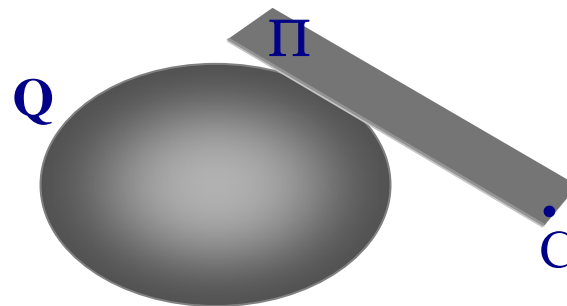
These lines are back-projected to planes $\mathbf{P}^T \mathbf{l}$ tangents to quadric \mathbf{Q} , satisfying:

$$\mathbf{\Pi}^T \mathbf{Q}^* \mathbf{\Pi} = 0$$

$$\mathbf{1}^T \mathbf{P} \mathbf{Q}^* \mathbf{P}^T \mathbf{1} = 0$$

Therefore:

$$\mathbf{C}^* = \mathbf{P} \mathbf{Q}^* \mathbf{P}^T$$

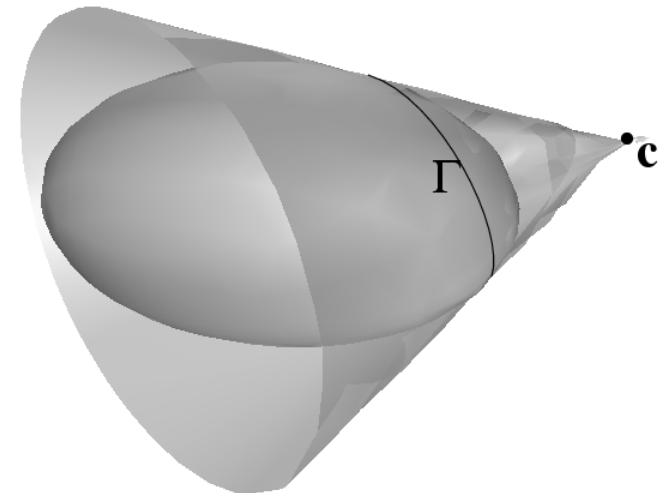


Action of Projective Camera on Quadrics

Forward Projection of Quadrics

$$C^* = PQ^*P^T$$

$$\Pi^T Q^* \Pi = 1^T P Q^* P^T 1 = 1^T C^* 1 = 0$$



The plane of Γ for a quadric Q and camera center C is given by $\Pi = QC$ (follows from pole-polar relation)

The cone with vertex C and tangent to the quadric Q is

$$Q_{C0} = (C^T Q C) Q - (Q C)(Q C)^T \quad Q_{C0} C = 0$$

Action of projective camera on quadrics

- Back-projection to cone: Example
 - The cone with vertex $C=(0,0,0,1)^T$ and tangent to the quadric Q :

$$Q = \begin{bmatrix} Q_{3 \times 3} & q \\ q^T & q_{44} \end{bmatrix}$$

$$Q_{co} = (C^T Q C) Q - (Q C)(Q C)^T$$

$$Q_{co} = \begin{bmatrix} q_{44} Q_{3 \times 3} - q q^T & 0 \\ 0 & 0 \end{bmatrix}$$

$$Q_{co} C = 0$$

