

3D Computer Vision Basic Transformations & 2D Projective Geometry

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based on a course given at UNC by Marc Pollefeys & the book "Multiple View Geometry" by Hartley & Zisserman

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chair for computer aided medical procedures

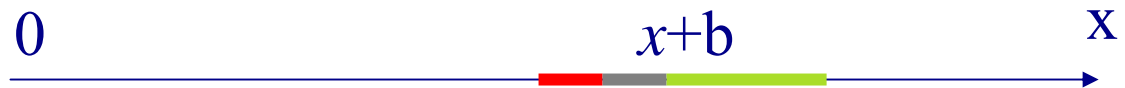
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1D Transformations in Euclidean Frame

1D Transformations



Euclidean transformation: translation by b



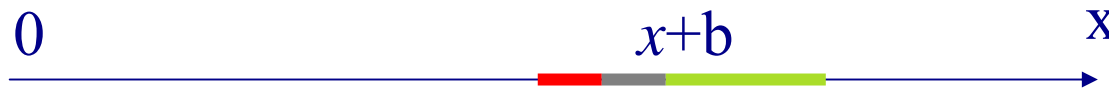
Similarity (Metric): translation and scaling



1D Transformations – Euclidean Transformation



Euclidean transformation: translation by b



Matrix-Vector representation: $x \rightarrow \begin{bmatrix} 1 & b \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$

Invariant: Distances (Lengths)

1D Transformations – Euclidean Transformation



Euclidean transformation: translation by b



Matrix-Vector representation:

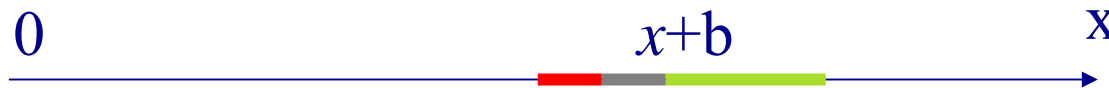
$$\begin{bmatrix} x \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

Invariant: Distances (Lengths)

1D Transformations – Euclidean Transformation



Euclidean transformation: translation by b



Matrix-Vector representation:

$$\begin{bmatrix} x \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

Invariant: Distances (Lengths)

1D Transformations – Euclidean Transformation



Euclidean transformation: translation by b



Homogeneous representation: $\lambda \begin{bmatrix} x \\ 1 \end{bmatrix} \rightarrow \lambda \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \rightarrow \textit{point } x + b$

Homogeneous representation $\begin{bmatrix} x \\ z \end{bmatrix} \rightarrow \textit{point } \frac{x}{z}$

1D Transformations – Metric Similarity



Similarity (Metric): translation and scaling



Matrix-Vector representation: $x \rightarrow \begin{bmatrix} a & b \\ x & 1 \end{bmatrix}$

Invariant: Ratio of distances

1D Transformations – Metric Similarity



Similarity (Metric): translation and scaling

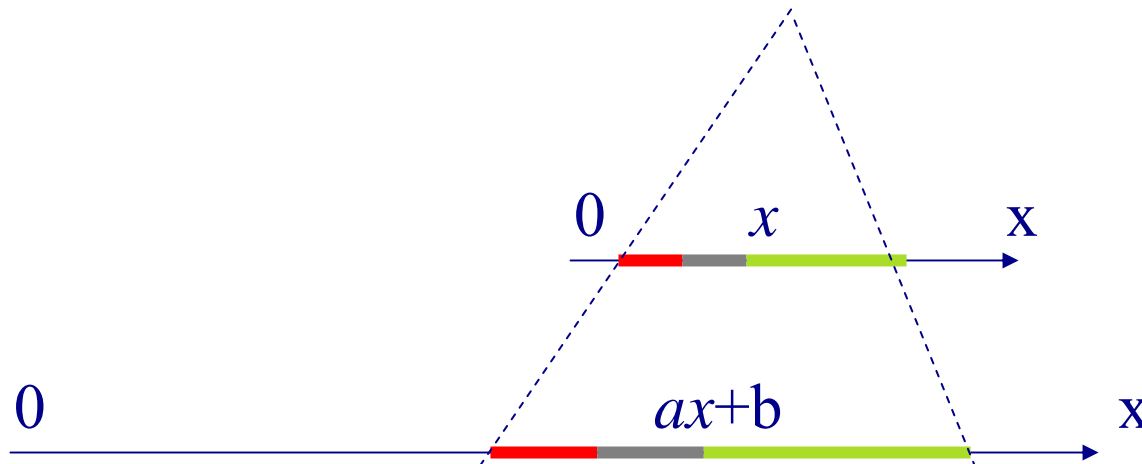


Matrix-Vector representation:

$$\begin{bmatrix} x \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

Invariant: Ratio of distances

1D Transformations – Metric Similarity

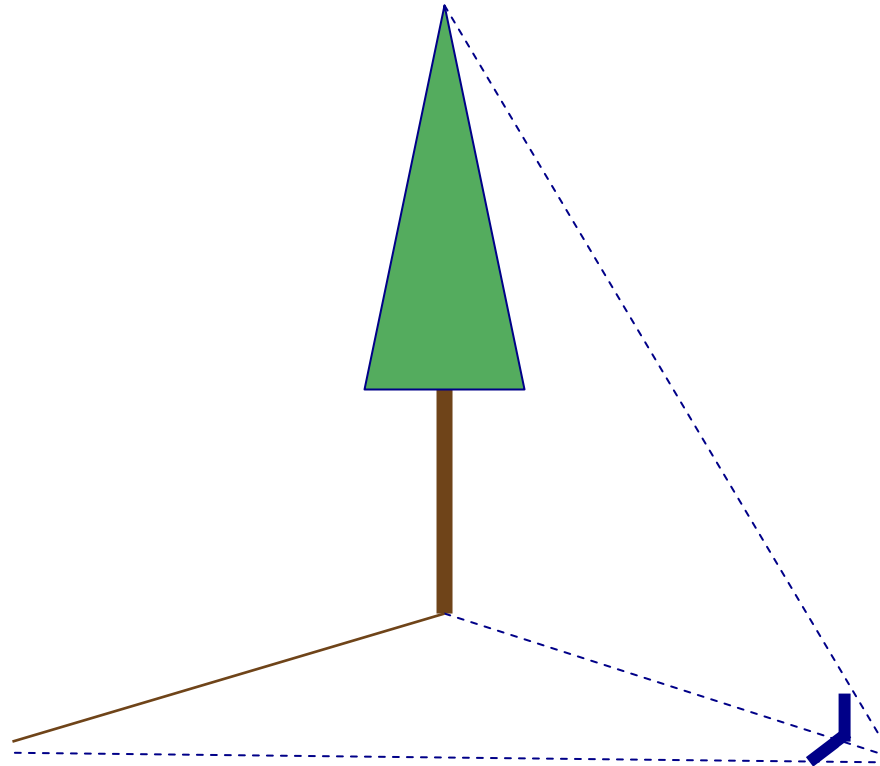


Matrix-Vector representation:

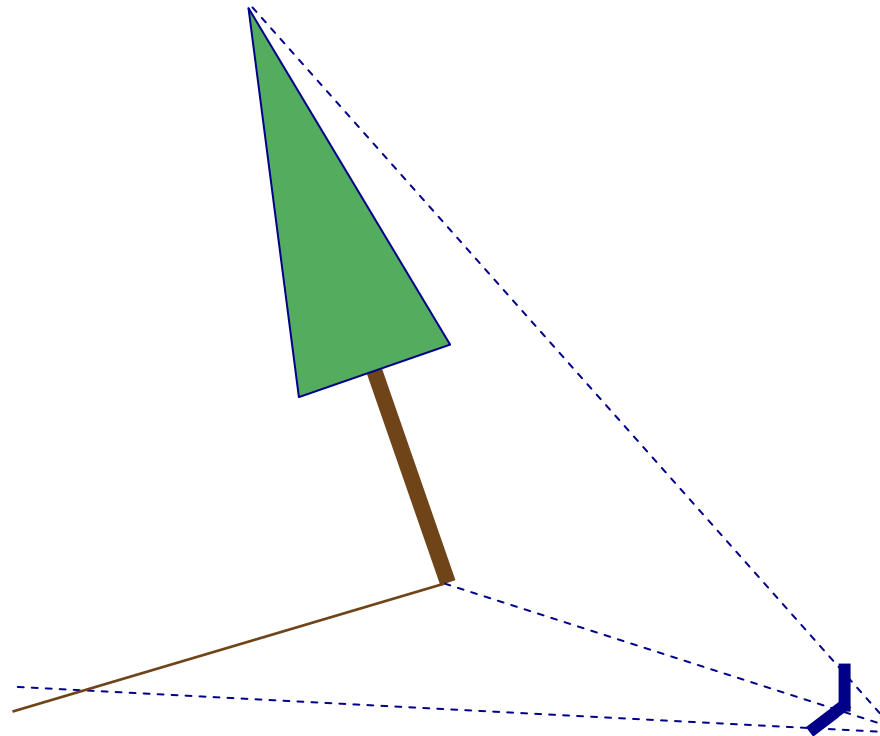
$$\begin{bmatrix} x \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

Invariant: Ratio of distances

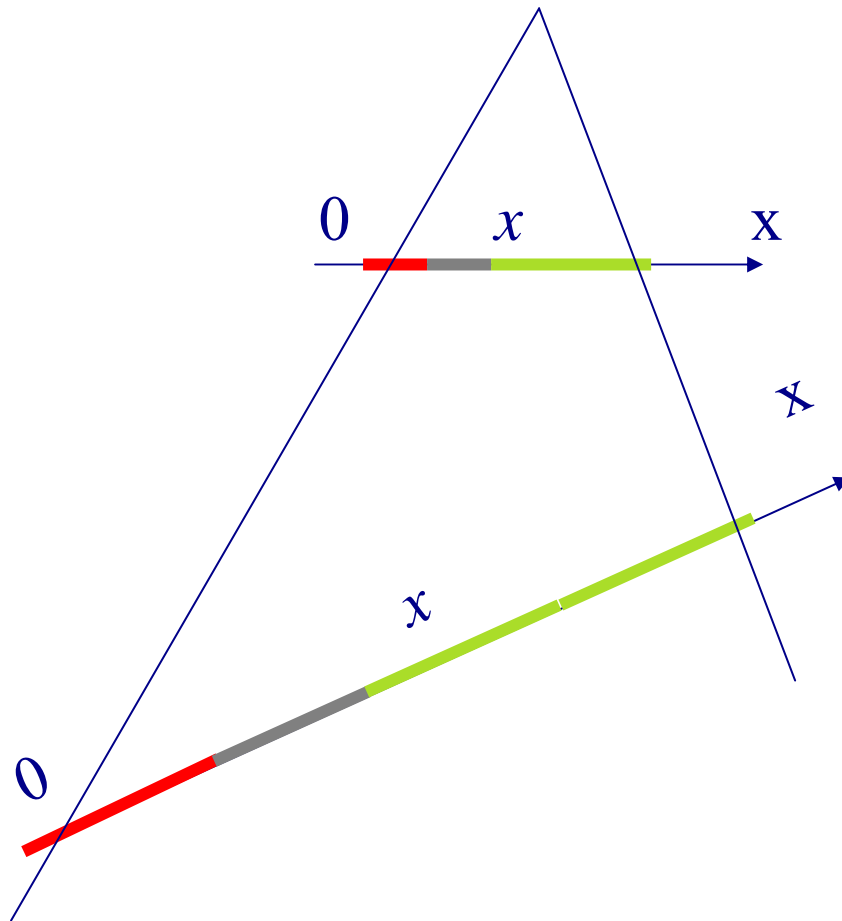
Measuring height of trees



Measuring height of “tilted” trees



1D Projective Transformation (P_1)



$$\begin{bmatrix} x \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ c & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

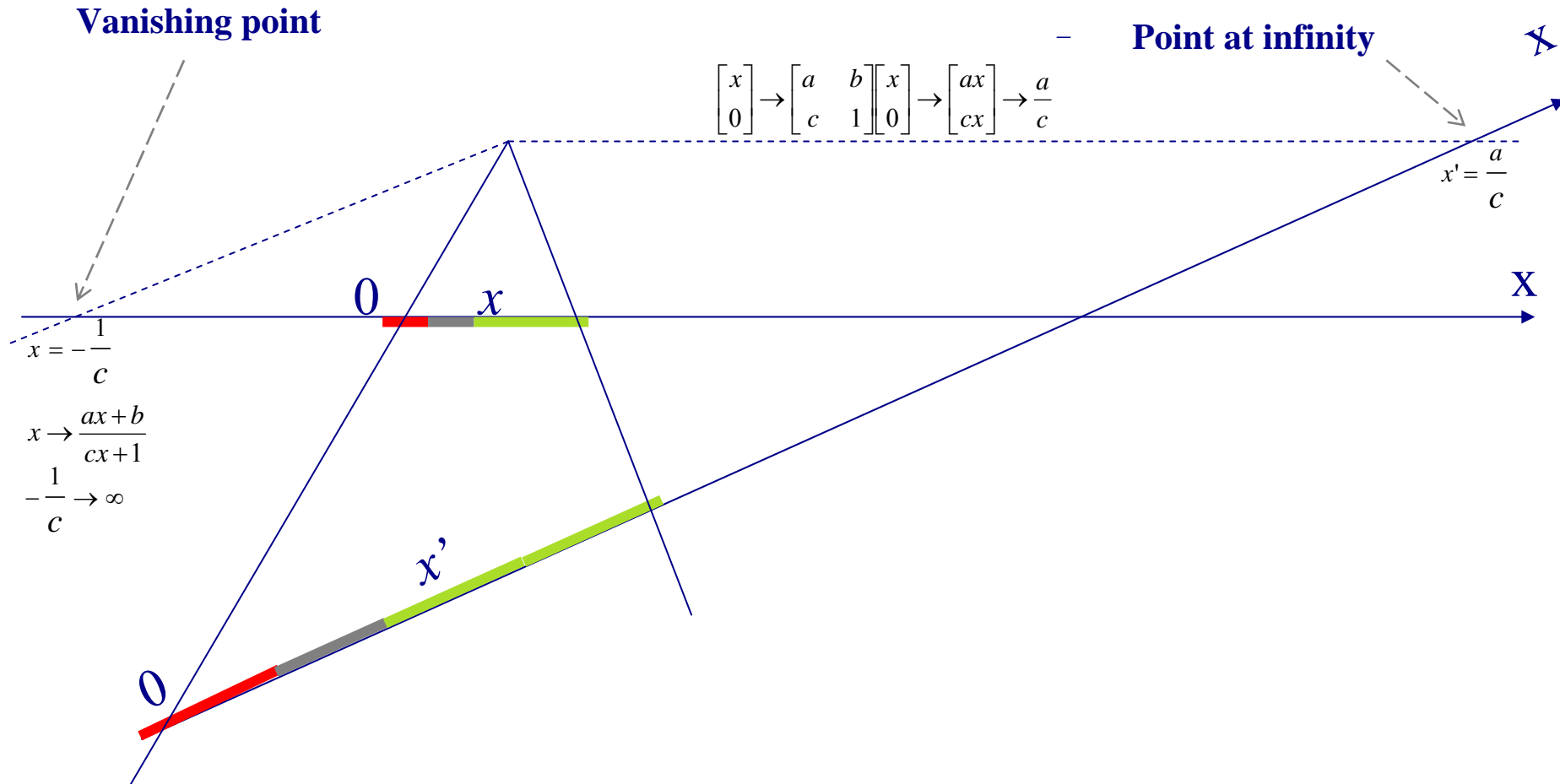
$$x \rightarrow \frac{ax + b}{cx + 1}$$

Point at Infinity

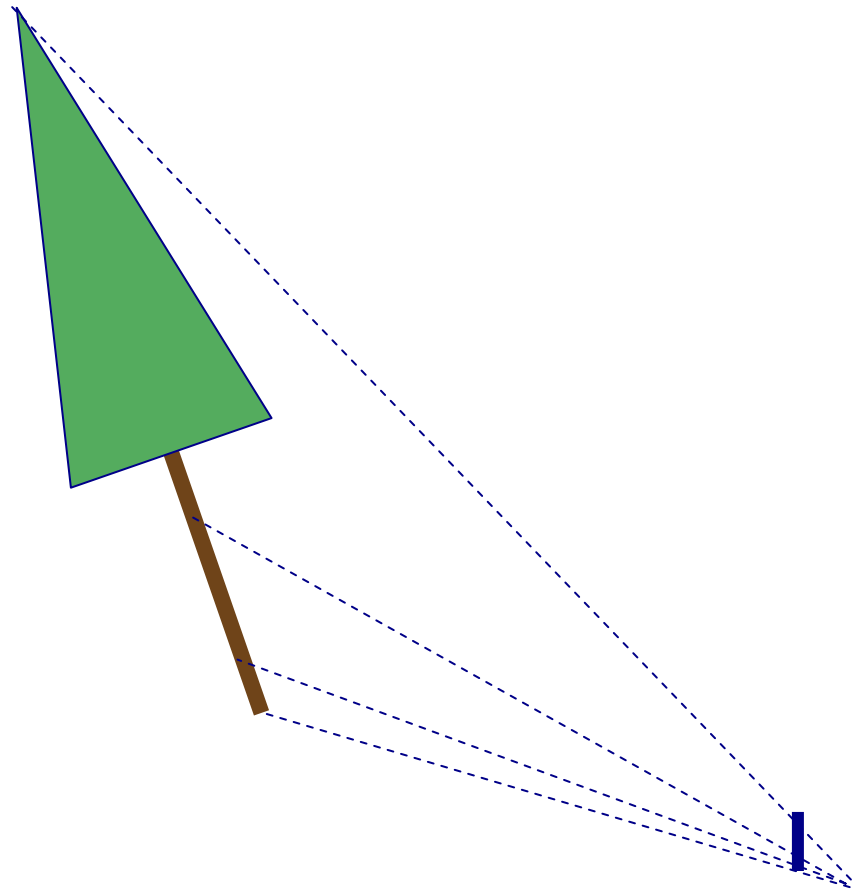
Homogeneous representation $\begin{bmatrix} x \\ z \end{bmatrix} \rightarrow \text{point } \frac{x}{z}$

$\begin{bmatrix} x \\ 0 \end{bmatrix} \rightarrow \text{point at Infinity}$

1D Projective Transformation (P_1)



Measuring height of “tilted” trees



Projective Geometry of 1D

Homogeneous point in 1D: $(x_1, x_2)^T$

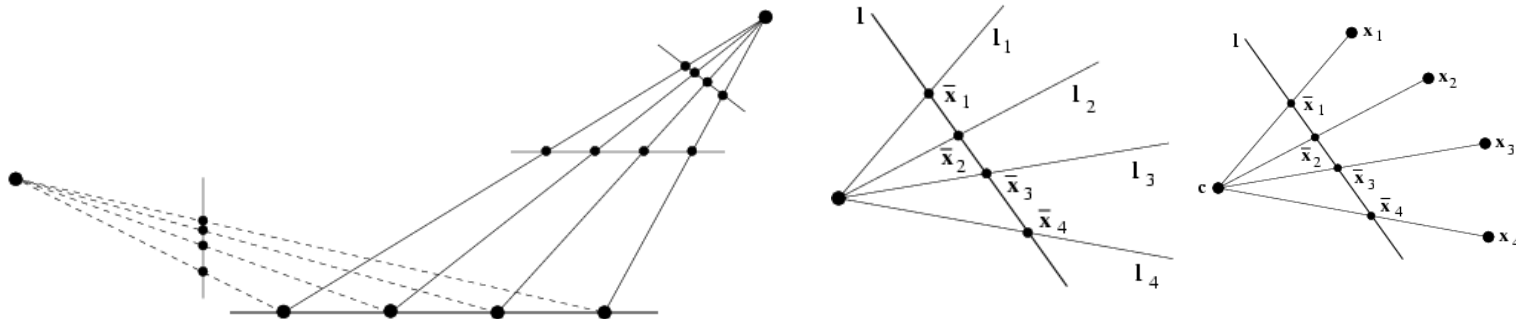
Point transformation in 1D: $\bar{x}' = \mathbf{H}_{2 \times 2} \bar{x}$ 3DOF (2x2-1)

The Cross Ratio

$$Cross(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) = \frac{|\bar{x}_1, \bar{x}_2| |\bar{x}_3, \bar{x}_4|}{|\bar{x}_1, \bar{x}_3| |\bar{x}_2, \bar{x}_4|}$$

$$|\bar{x}_i, \bar{x}_j| = \det \begin{bmatrix} x_{i1} & x_{j1} \\ x_{i2} & x_{j2} \end{bmatrix}$$

The Cross Ratio remains invariant under projective transformations

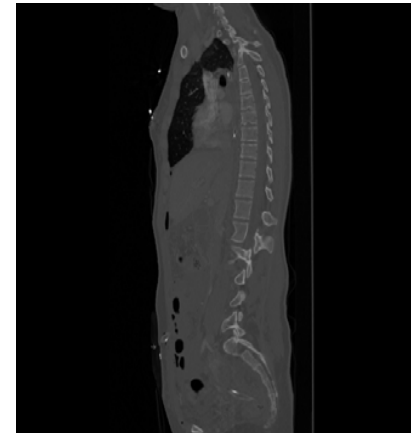
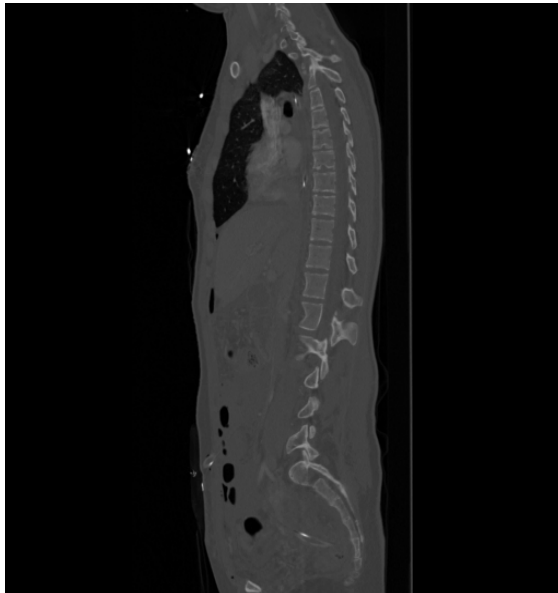


2D Transformations

2D Transformations

- Scaling
 - Isotropic Scaling
 - Non-isotropic Scaling
- Rotation
- Translation
- Euclidean Transformation
- Metric Transformation (Similarity)
- Affine Transformation
- Projective Transformation

2D Transformations – Scaling (isotropic)

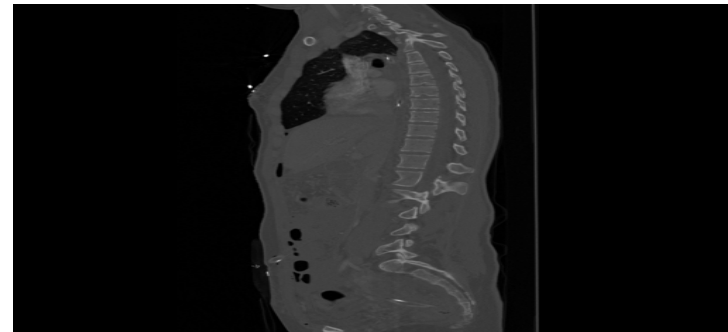
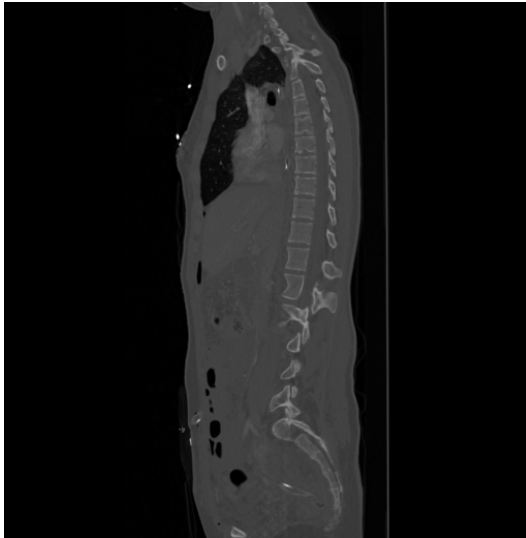


2D Transformations – Isotropic Scaling

- same scaling factor in both dimensions

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} sx \\ sy \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Transformations – Non-Isotropic Scaling

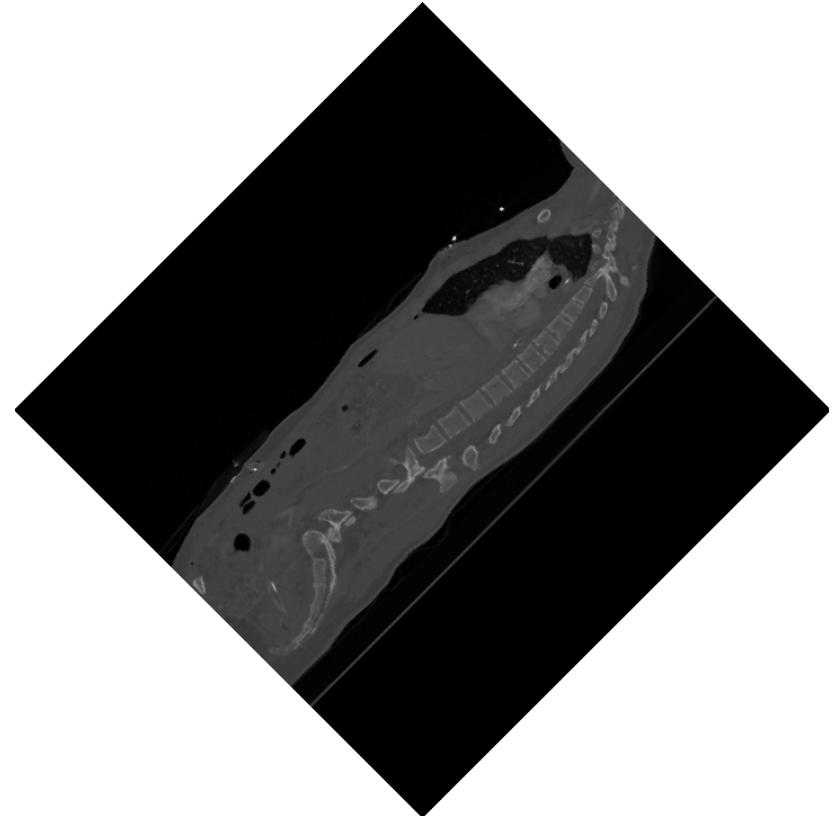
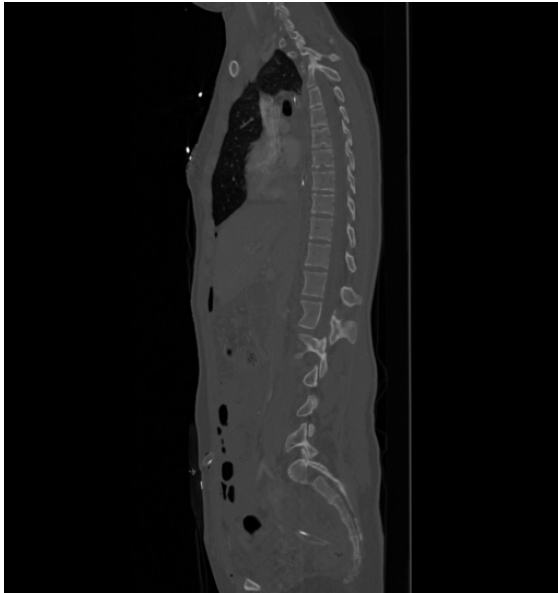


2D Transformations – Non-Isotropic Scaling

- different scaling factors in the two dimensions

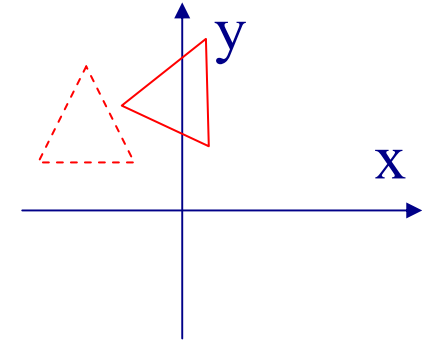
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_1 x \\ s_2 y \end{bmatrix} = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Transformations - Rotation



2D Transformations - Rotation

Rotation around origin by angle θ



Rotation Matrix

$$R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Transformed Point

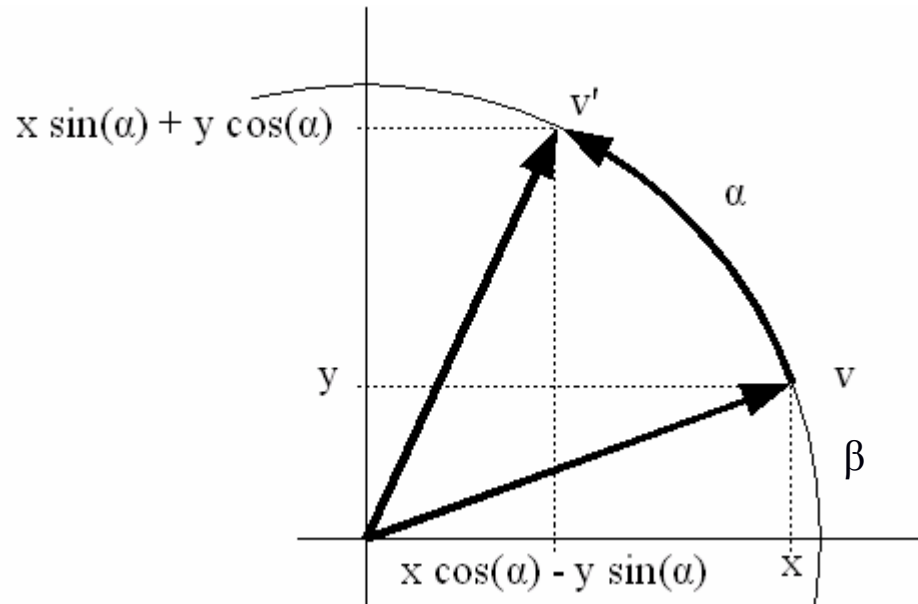
$$\vec{p}' = R\vec{p}$$

where

$$x' = x \cdot \cos \theta - y \cdot \sin \theta$$

$$y' = x \cdot \sin \theta + y \cdot \cos \theta$$

2D Transformations - Rotation



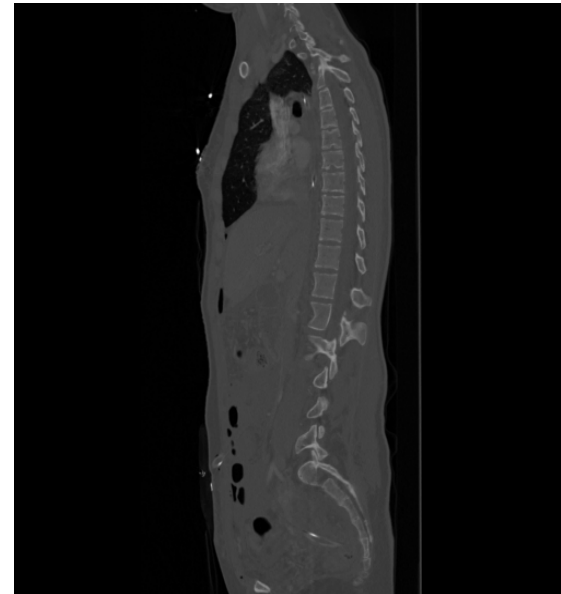
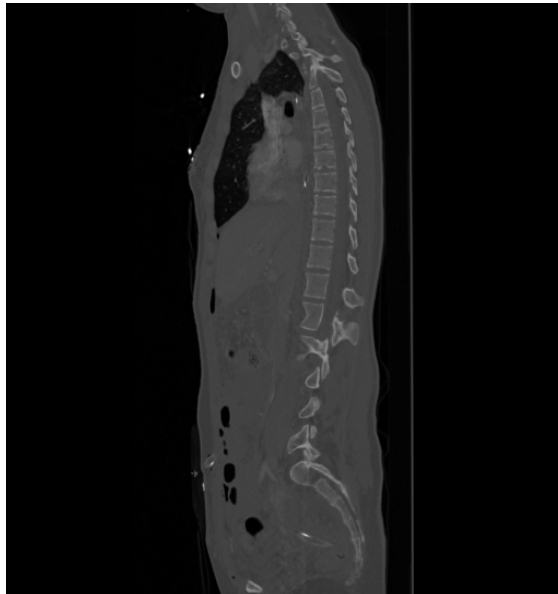
$$x' = v * \cos(\alpha + \beta) = v * \cos(\alpha) * \cos(\beta) - v * \sin(\alpha) * \sin(\beta)$$

$$x' = x \cdot \cos \alpha - y \cdot \sin \alpha$$

$$y' = v * \sin(\alpha + \beta) = v * \cos(\alpha) * \sin(\beta) + v * \sin(\alpha) * \cos(\beta)$$

$$y' = x \cdot \sin \alpha + y \cdot \cos \alpha$$

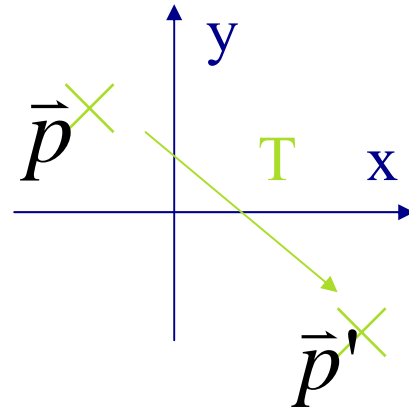
2D Transformations - Translation



2D Transformations - Translation

Original point $\vec{p} = \begin{bmatrix} x \\ y \end{bmatrix}$

Transformed point $\vec{p}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$



Translation Transformation $\vec{p}' = \vec{p} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$

2D Transformations - Translation

- Homogeneous Coordinates:

How to put the translation transformation $\vec{p}' = \vec{p} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$ into the *Matrix x Vector* form $p'=Ap$?

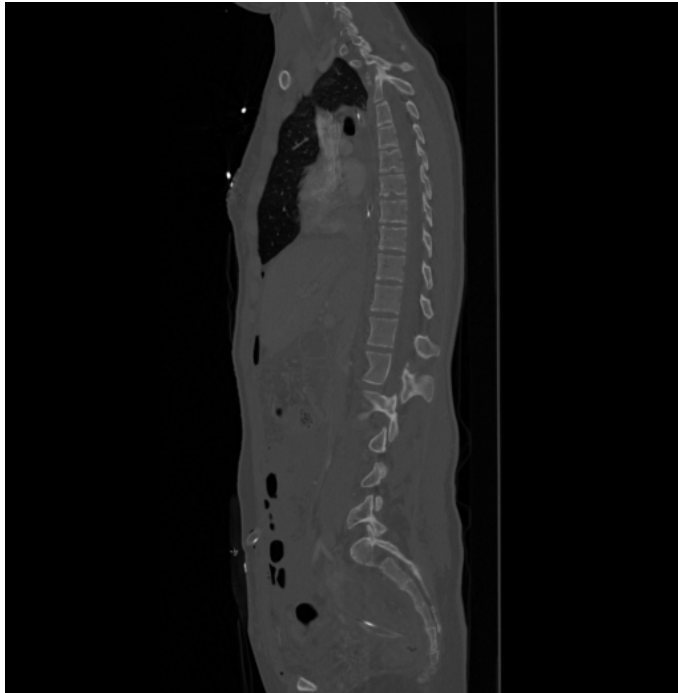
Homogeneous point $\vec{p} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

Transformation Matrix $\vec{p}' = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+t_x \\ y+t_y \\ 1 \end{bmatrix}$

Homogeneous coordinates of Points:

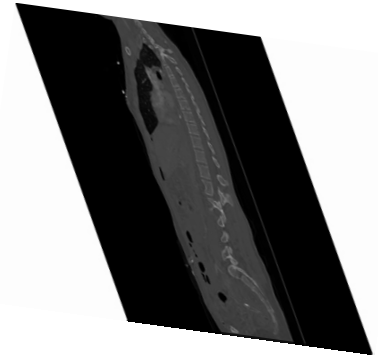
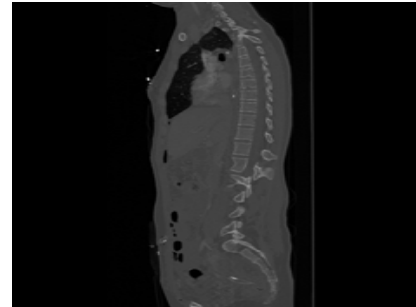
$$\text{real point } [x, y] \text{ (homogenous representation)} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda \end{bmatrix}$$

2D Transformations – Affine Transformation



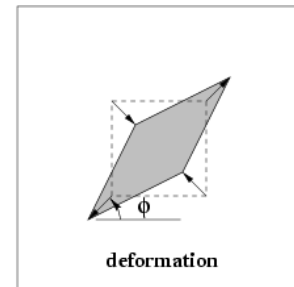
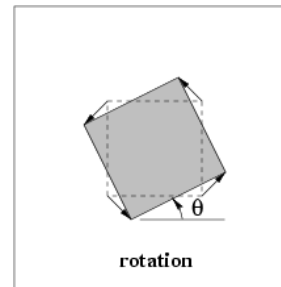
2D Transformations – Affine Transformation

$$\mathbf{x}' = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x} = \mathbf{A}\mathbf{x}$$

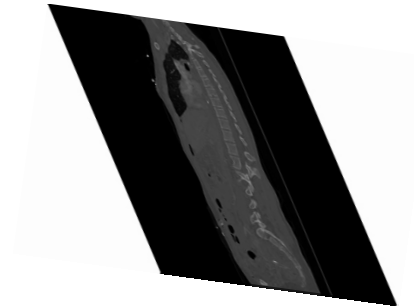
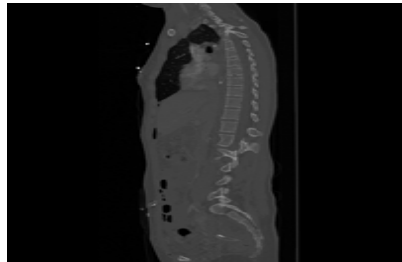


$$\mathbf{A} = \mathbf{R}(\theta) \cdot \mathbf{R}(-\phi) \cdot \mathbf{D} \cdot \mathbf{R}(\phi)$$

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$



Affine Transformation: Homogenous coordinate



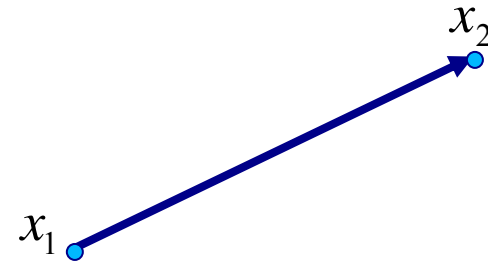
$$\mathbf{x}' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} ax + by \\ cx + dy \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathbf{A}\mathbf{x}$$

Homogeneous coordinates of Points:

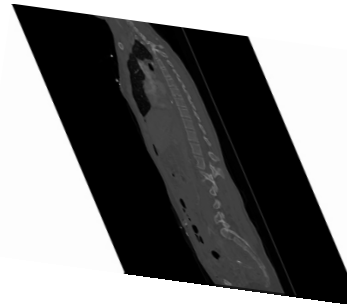
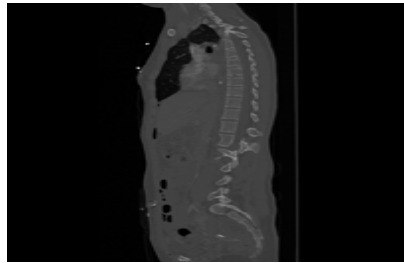
real point $[x, y]$ (*homogenous representation*) \rightarrow
$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda \end{bmatrix}$$

points at infinity :
$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

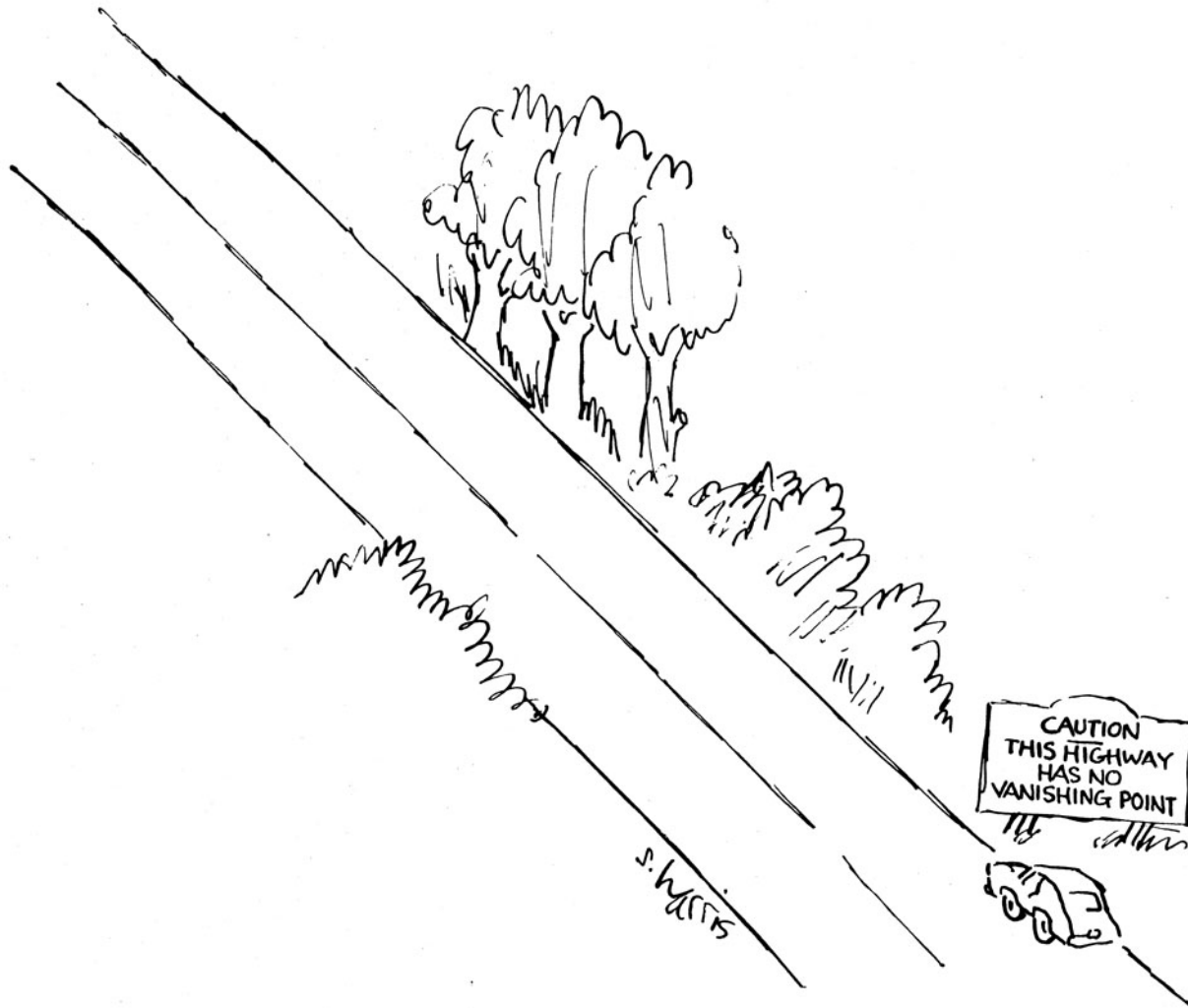
example :
$$\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} - \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ 0 \end{bmatrix}$$



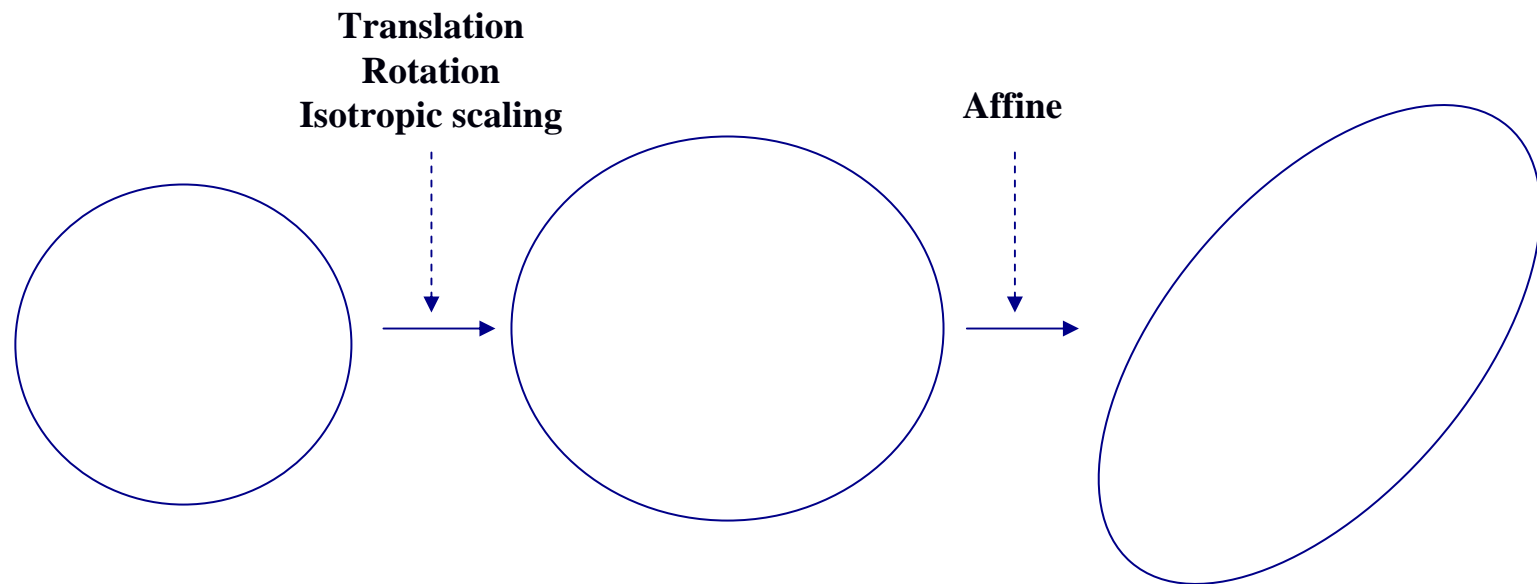
Affine Transformation: Points at Infinity



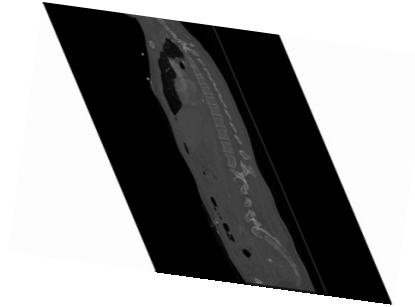
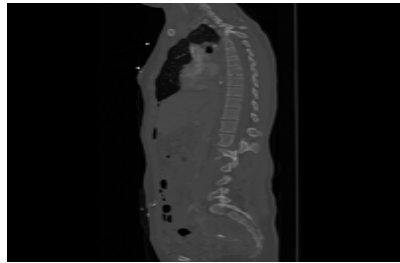
$$\mathbf{x}' = \begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} ax + by \\ cx + dy \\ 0 \end{bmatrix}$$



Affine Transformation: Conics

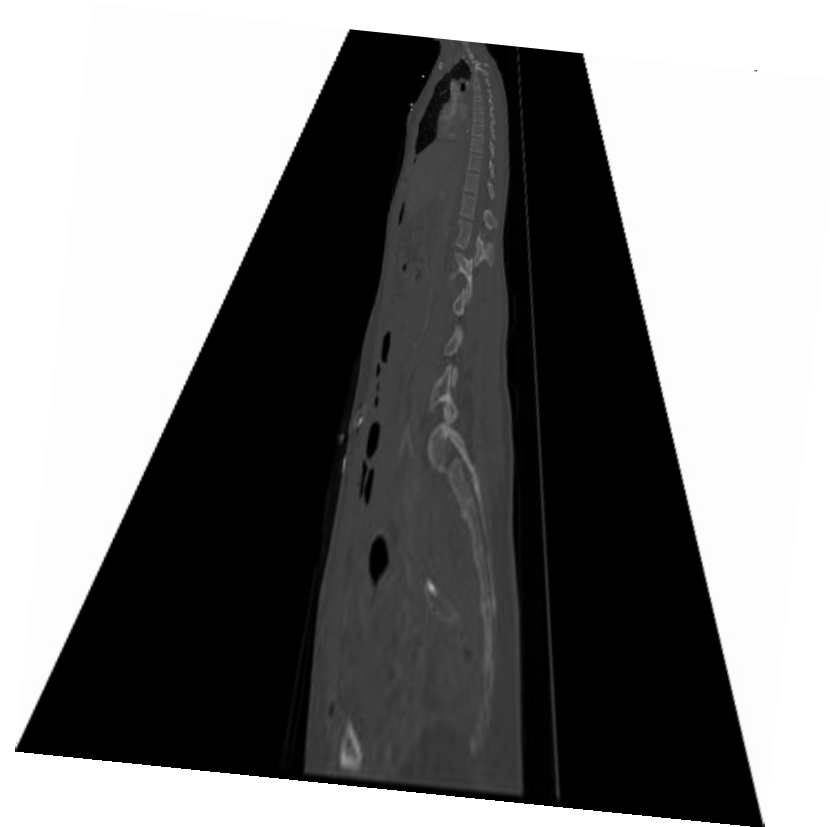
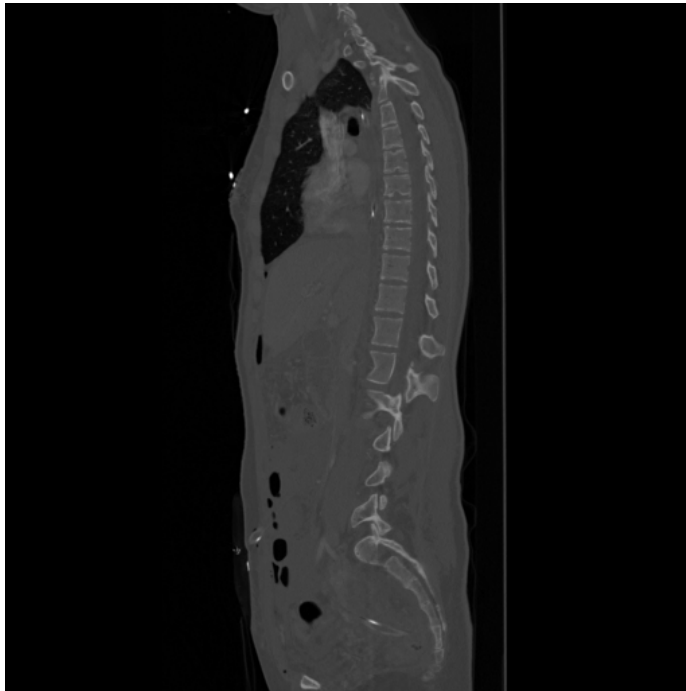


General Affine Transformation (6 parameters)

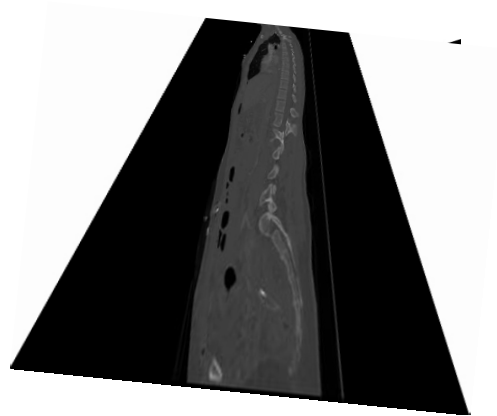
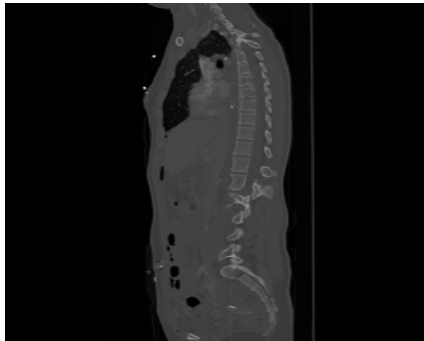


$$\mathbf{x}' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathbf{A}\mathbf{x}$$

2D Transformations – Projective Transformations

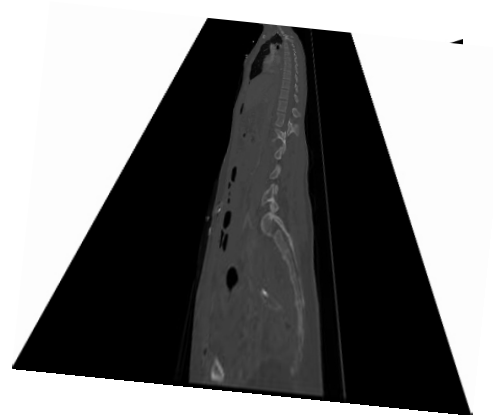
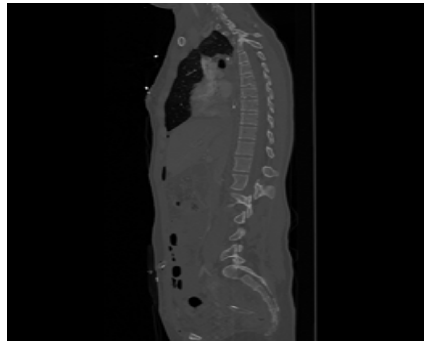


2D Transformations – Projective Transformations



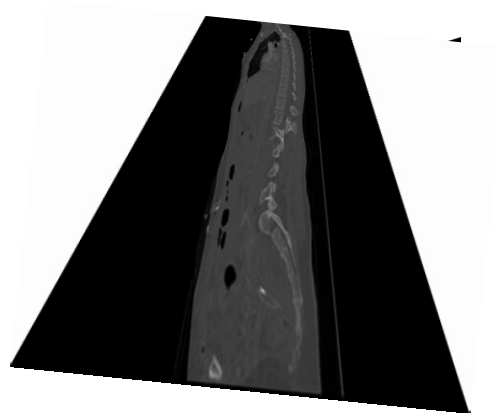
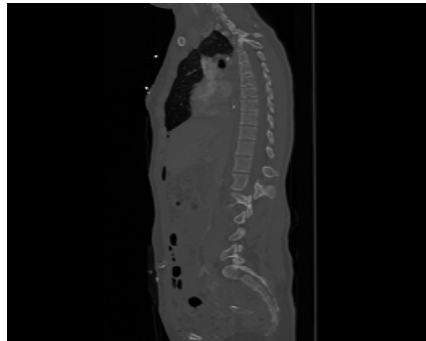
$$\mathbf{x}' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} ax + by + c \\ dx + ey + f \\ gx + hy + 1 \end{bmatrix} = \lambda \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathbf{H}\mathbf{x}$$

2D Transformations – Projective Transformations



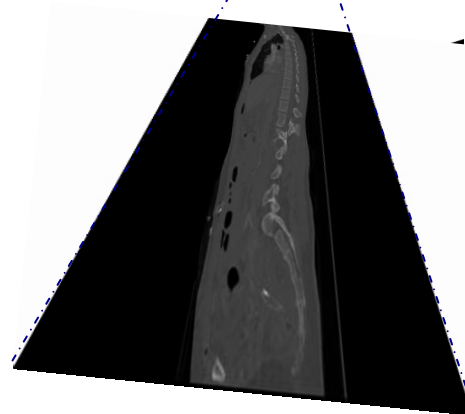
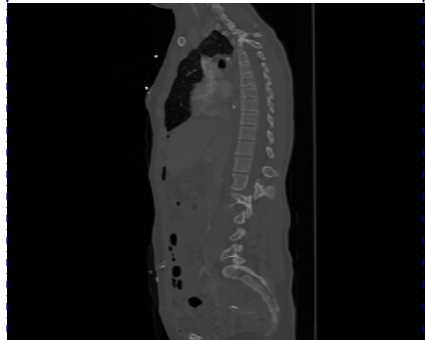
$$\mathbf{x}' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} ax + by + c \\ dx + ey + f \\ gx + hy + 1 \end{bmatrix} \rightarrow \begin{cases} x' = \frac{ax + by + c}{gx + hy + 1} \\ y' = \frac{dx + ey + f}{gx + hy + 1} \end{cases}$$

Affine Transformation: Points at Infinity



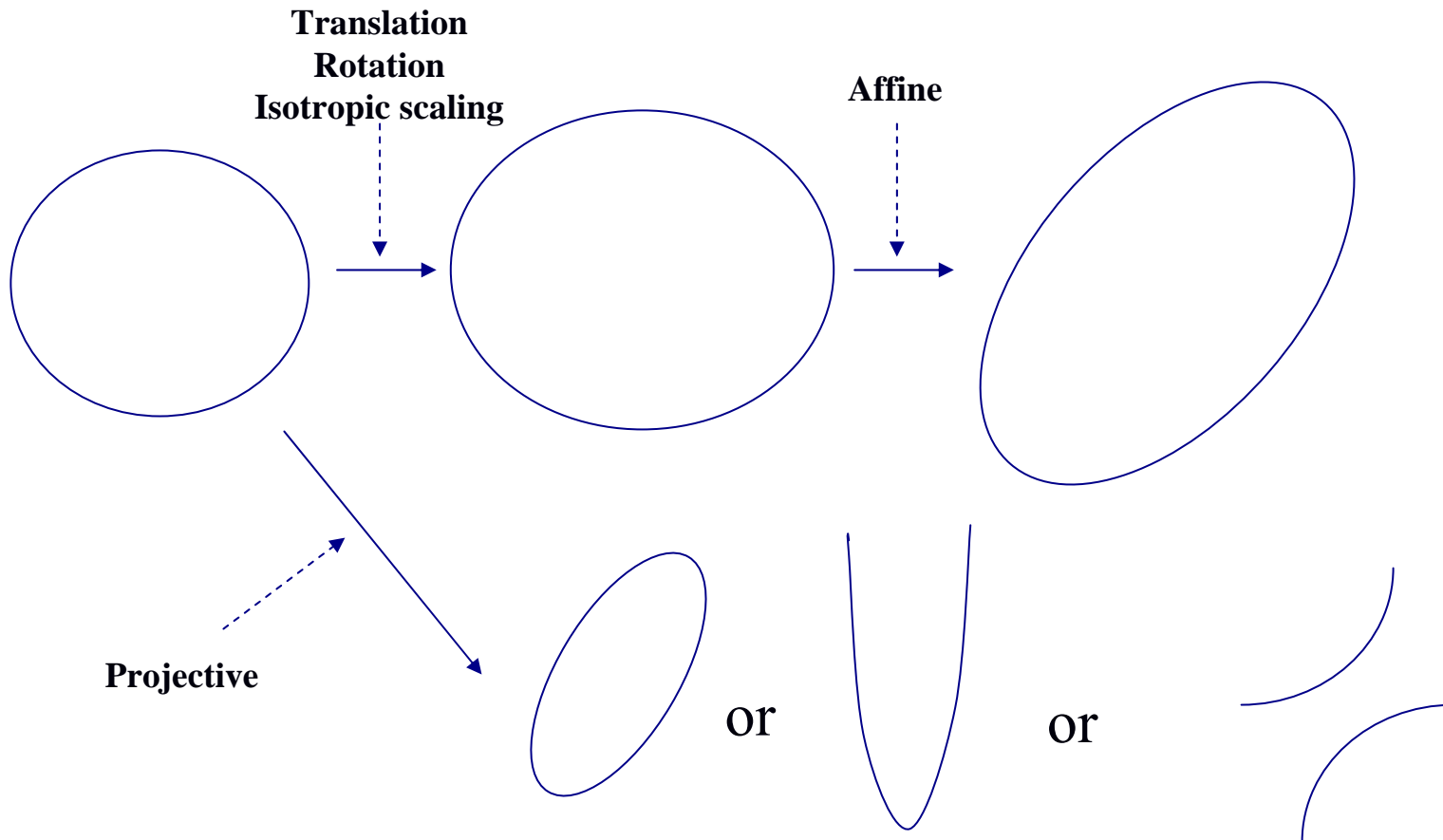
$$\mathbf{x}' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} ax + by \\ dx + ey \\ gx + hy \end{bmatrix}$$

2D Transformations – Projective Transformations



$$\mathbf{x}' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \mathbf{H} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} ax + by \\ dx + ey \\ gx + hy \end{bmatrix} \rightarrow \begin{cases} x' = \frac{ax + by}{gx + hy} \\ y' = \frac{dx + ey}{gx + hy} \end{cases}$$

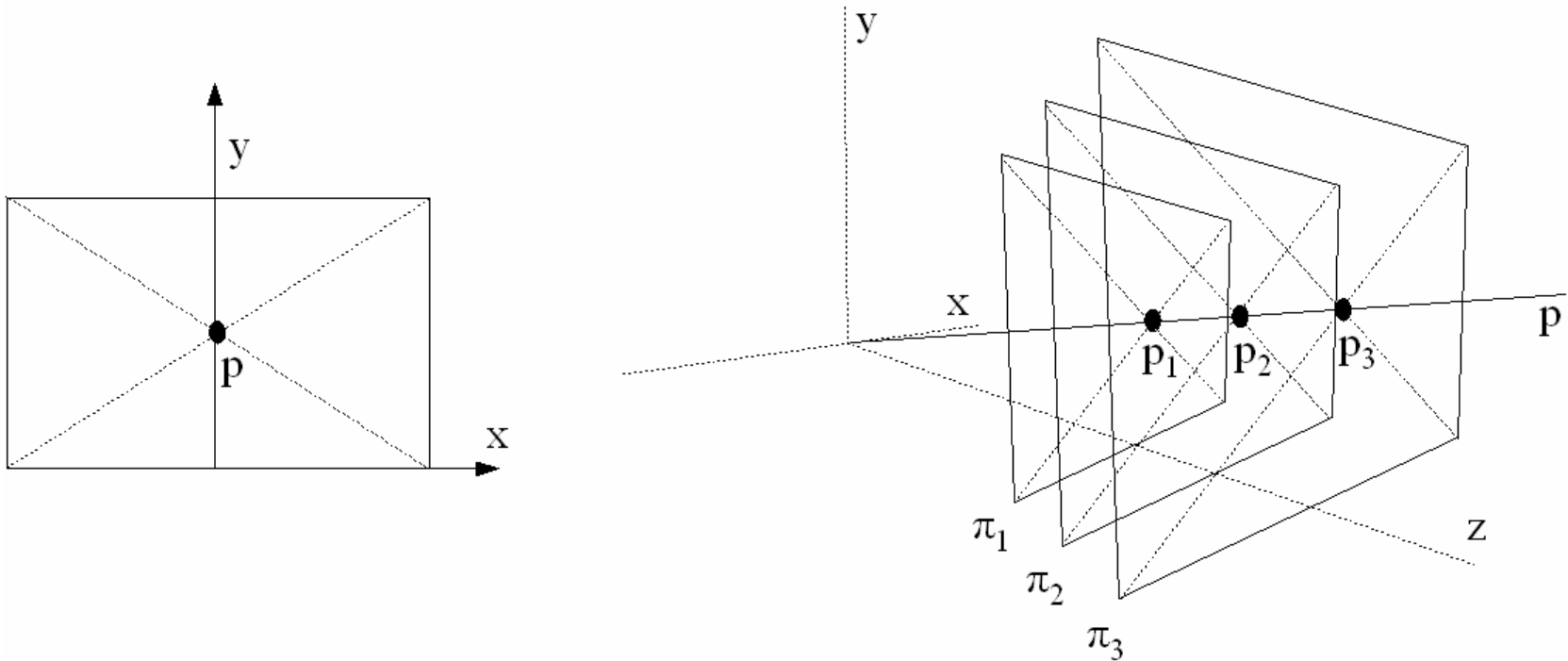
Projective Transformation: Conics



2D Projective Space – The Projective Plane

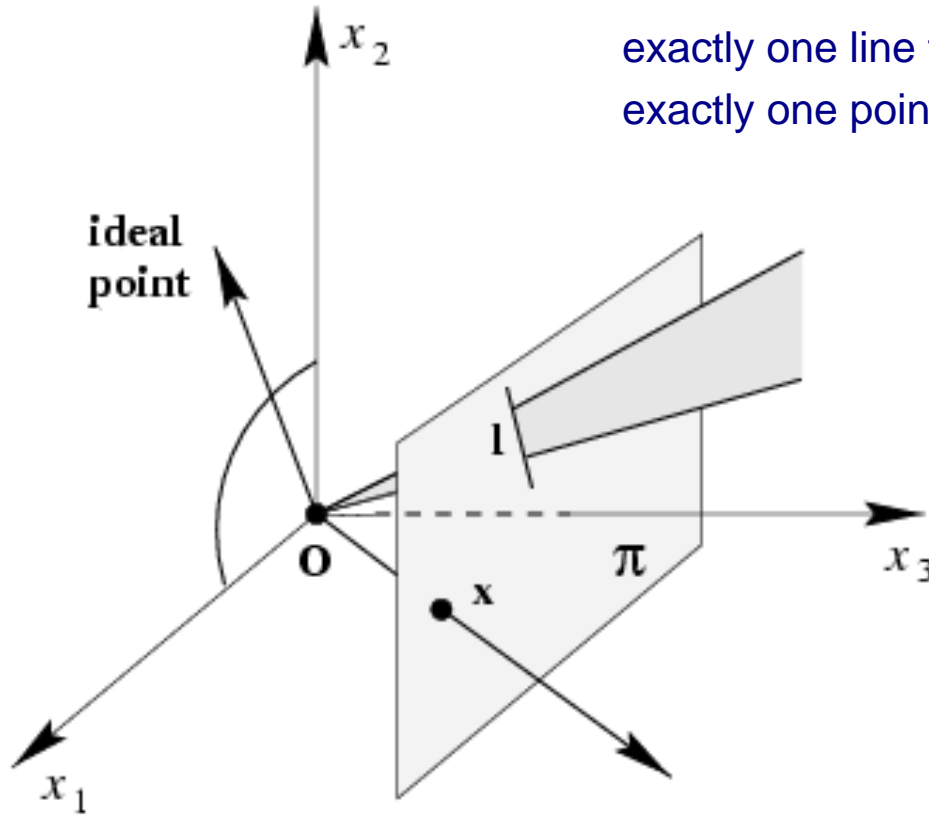
- A line in plane can be represented as: $ax+by+c=0$
- We can represent any line l by three parameters a , b , and c
- The same line l can be represented by multiples of a , b and c (ka , kb and kc for a scalar k) since: $kax+kby+kc=k*0=0=ax+by+c$
- A line is represented by a set of vectors, which differ only in scale k (k non zero)
- This equivalence set of vectors is known as a **Homogeneous Vector**
- Any particular vector of the class is a representative of the equivalence class
- The set of all these equivalence classes (i.e. homogeneous vectors) of vectors in \mathbb{R}^3 (excluding $(0,0,0)$) is called the **Projective Space P^2**

A Model for the Projective Plane



We'll use this intuition throughout the lesson!!!

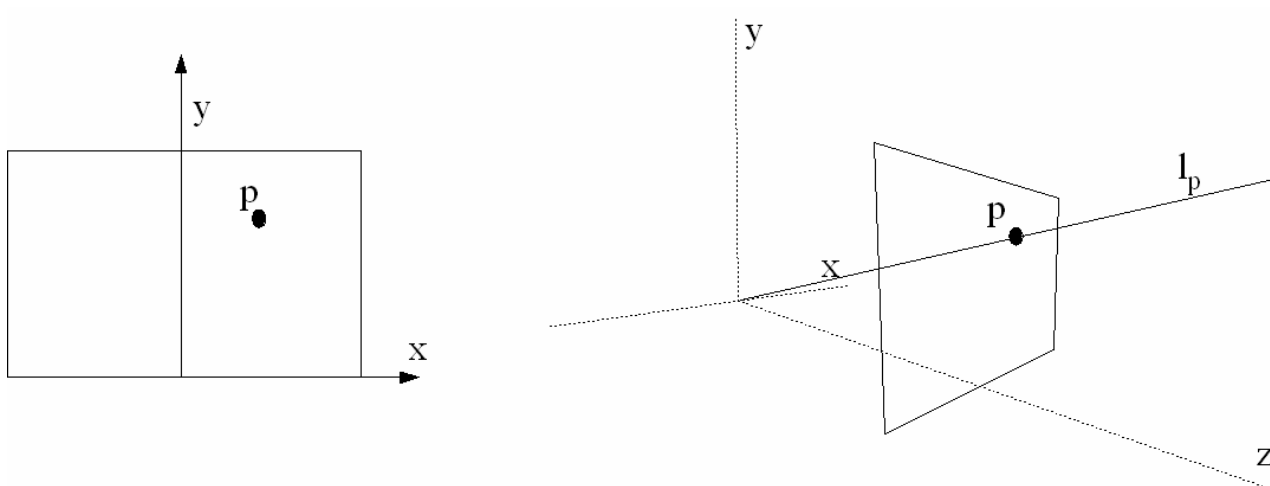
A Model for the Projective Plane



exactly one line through two points

exactly one point at intersection of two lines

Homogeneous Coordinates - Points



$$p = (x, y, 1)^T \sim k(x, y, 1)^T, \forall k \neq 0$$

- a point is represented in P^2 by a simple vector
- remember that every vector in P^2 is just a representative of an equivalence class of vectors
- hence a point is represented by any vector of the equivalence class

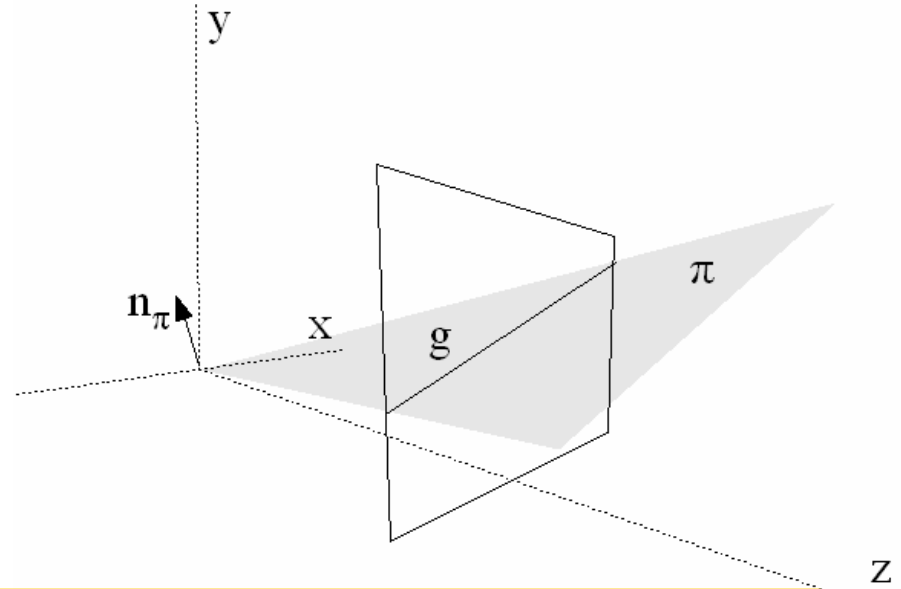
Homogeneous Coordinates - Lines

- In P^2 lines as well are represented as vectors
- because of the equivalence class of vectors, any vector of the class is representative

$$ax + by + c = 0 \quad (a, b, c)^T$$

$$(ka)x + (kb)y + kc = 0, \forall k \neq 0$$

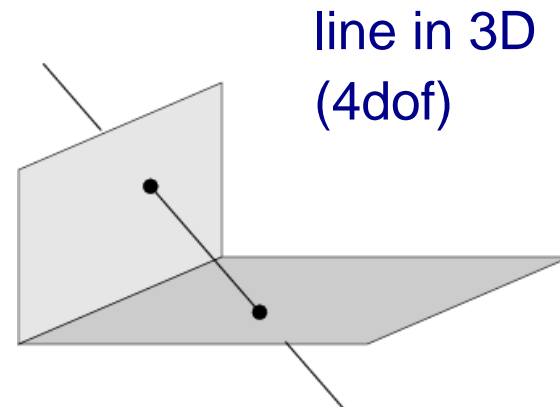
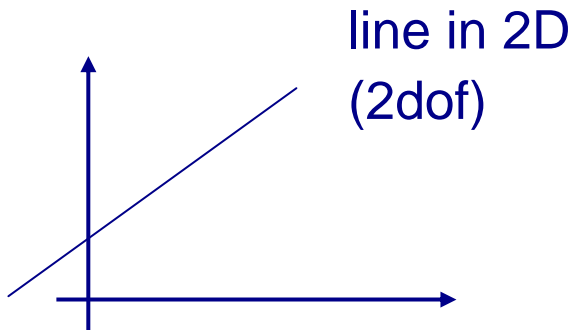
$$(a, b, c)^T \sim k(a, b, c)^T$$



Note that Points and Lines are represented both in P^2 as vectors

Degrees of Freedom (DOF)

- DOF is the number of parameters needed to specify an object in the respective space
- examples:



Homogeneous vectors (in P^2) have 2 DOF

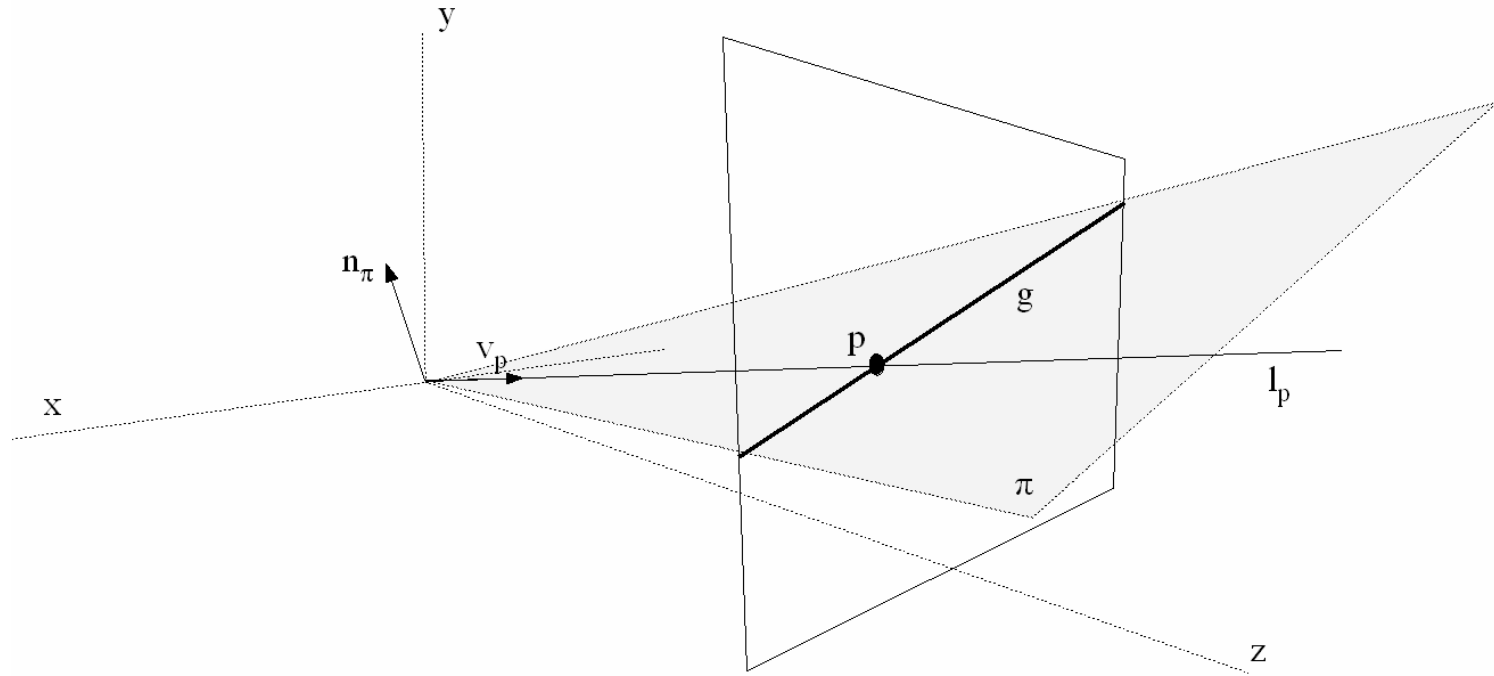
Any homogeneous vector x (in P^2) has 3 coordinates but only 2 DOF since it is up to scale – hence only the two ratios $x_1:x_3$ and $x_2:x_3$ between the coordinates are significant.

Homogeneous vector x : $(x_1, x_2, x_3)^T = k \cdot (x_1, x_2, x_3)^T$

The ratios $x_1:x_3$ and $x_2:x_3$:

$$x_1 : x_3 = \frac{x_1}{x_3} = \frac{kx_1}{kx_3} \quad x_2 : x_3 = \frac{x_2}{x_3} = \frac{kx_2}{kx_3}$$

Incidence of Points and Lines



The point x lies on the line l if and only if $x^T l = l^T x = 0$

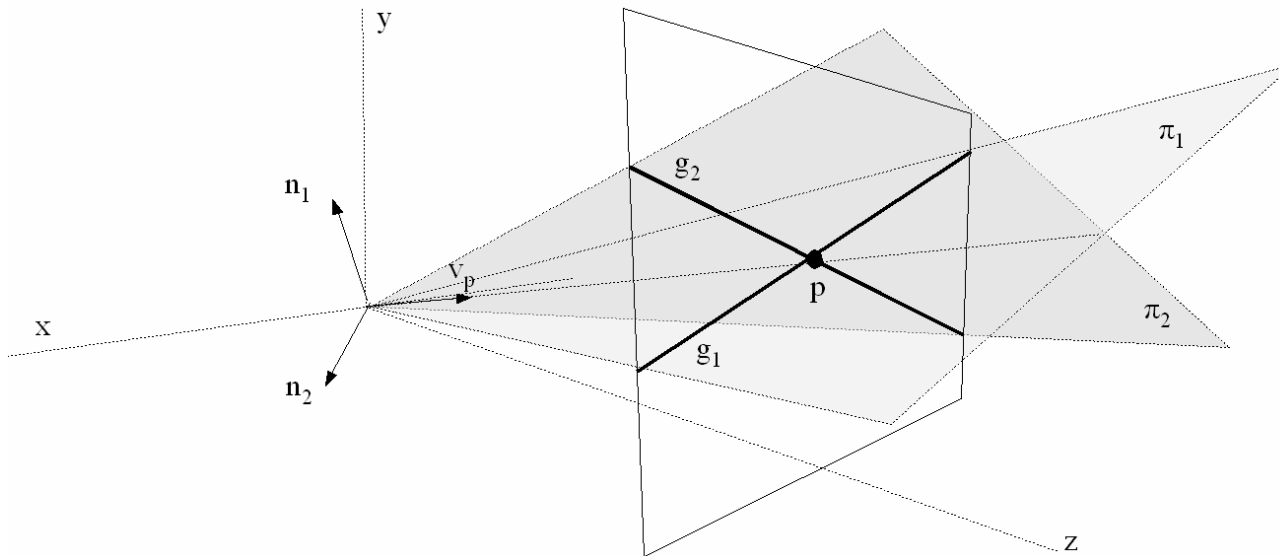
$$x = (x, y, 1)^T \text{ on } l = (a, b, c)^T \text{ if and only if } ax + by + c = 0$$

$$(x, y, 1)(a, b, c)^T = x^T l = 0$$

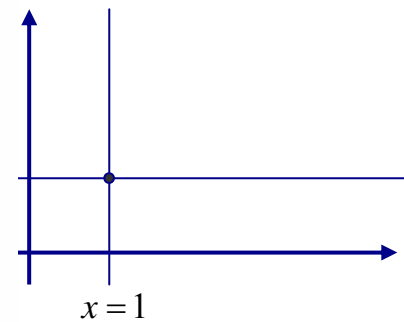
Points from Lines - Intersections of Lines

The intersection point \mathbf{x} of two lines \mathbf{l}_1 and \mathbf{l}_2 satisfies :

$$\mathbf{x}^t \mathbf{l}_1 = \mathbf{x}^t \mathbf{l}_2 = 0 \rightarrow \mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2$$



Example

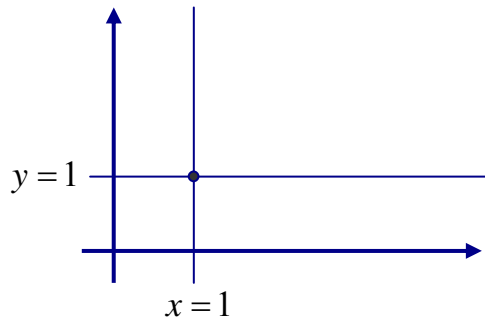


Points from Lines - Intersections of Lines

The intersection point \mathbf{x} of two lines \mathbf{l}_1 and \mathbf{l}_2 satisfies :

$$\mathbf{x}^t \mathbf{l}_1 = \mathbf{x}^t \mathbf{l}_2 = 0 \rightarrow \mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2 \rightarrow \mathbf{x} = -\mathbf{l}_2 \times \mathbf{l}_1$$

Example



$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \approx \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Alternative Notation for the Cross Product

The cross product of two vectors **a** and **b** (**a**×**b**) can be written as a product of an anti-symmetric matrix **[a]_x** and the vector **b**

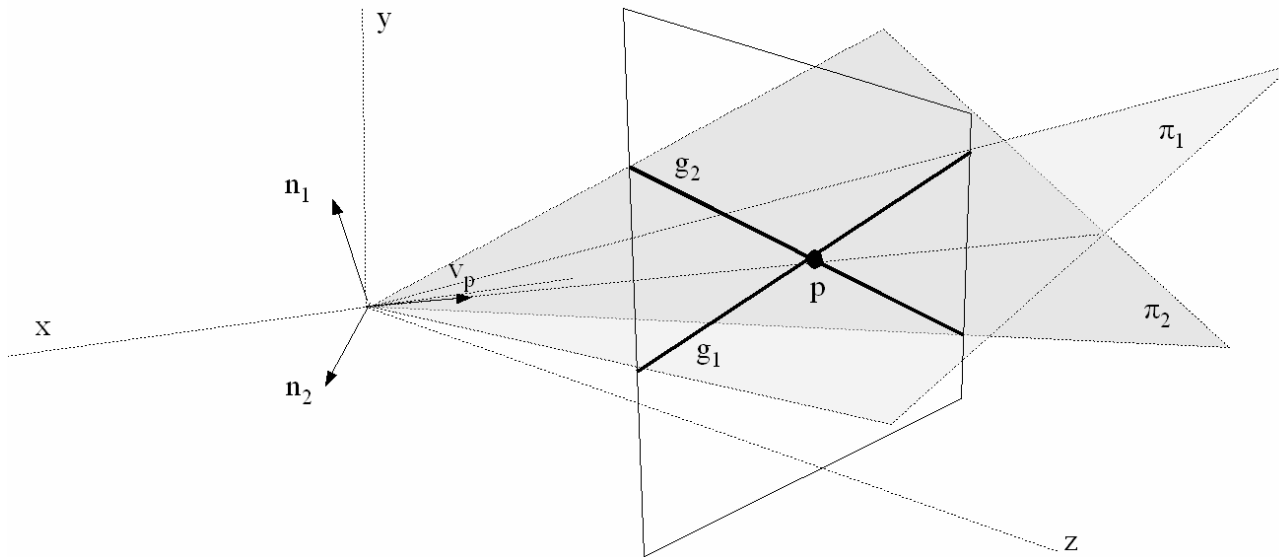
$$a \times b = [a]_x b = -[b]_x a$$

The appropriate anti-symmetric matrix **[a]_x** is:

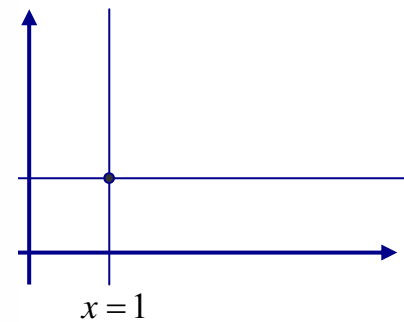
$$[a]_x = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

Points from Lines - Intersections of Lines

The intersection of two lines l and l' is the point $x = l \times l' = [l]_x l' = -[l']_x l$



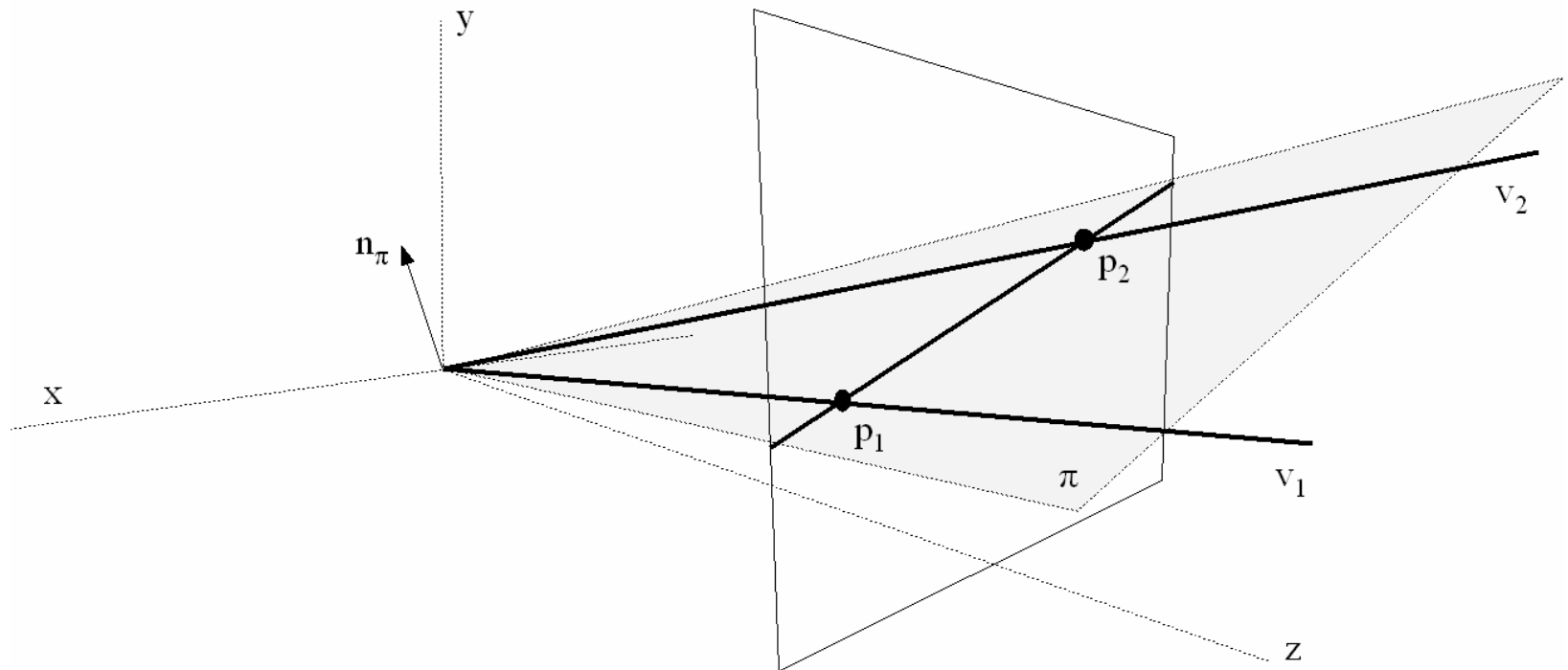
Example



Lines from Points - Line joining two Points

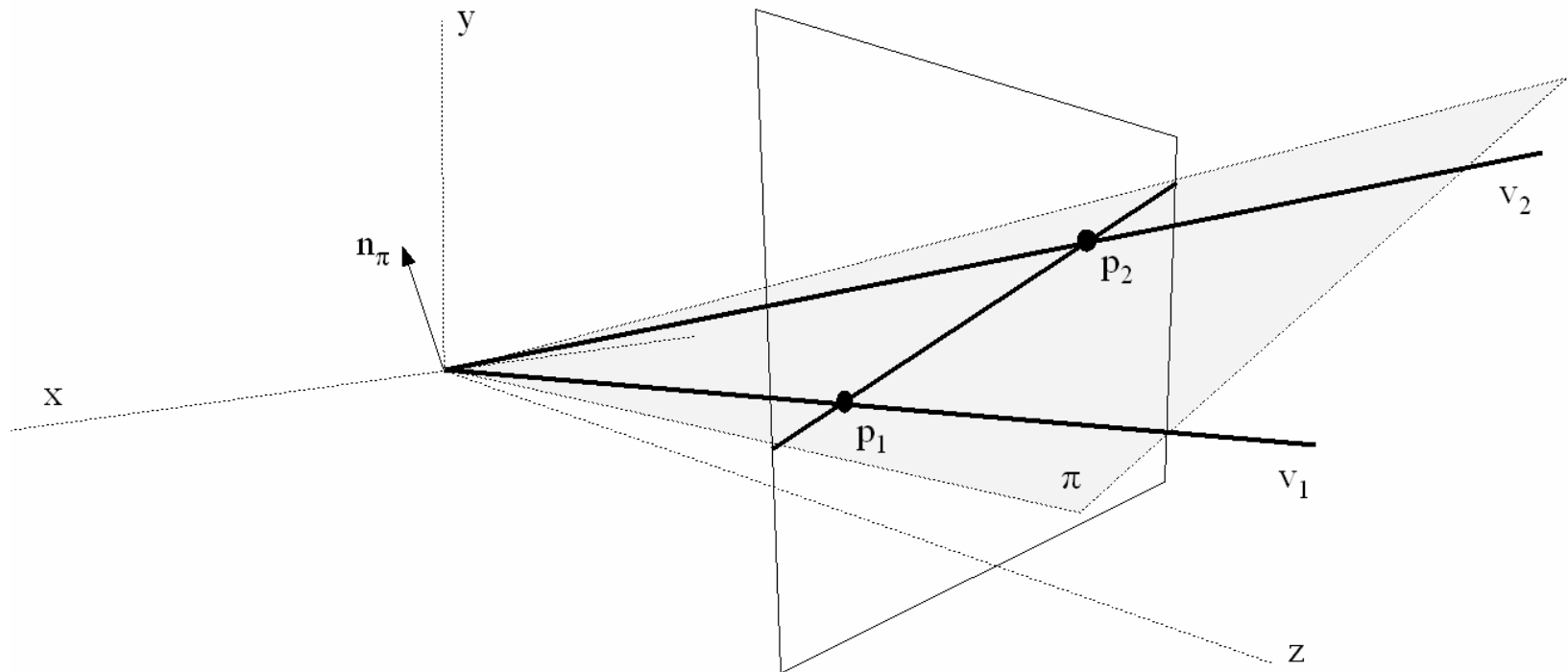
The line \mathbf{l} joining two points \mathbf{x}_1 and \mathbf{x}_2 satisfies :

$$\mathbf{x}_1^t \mathbf{l} = \mathbf{x}_2^t \mathbf{l} = 0 \rightarrow \mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2$$



Lines from Points - Line joining two Points

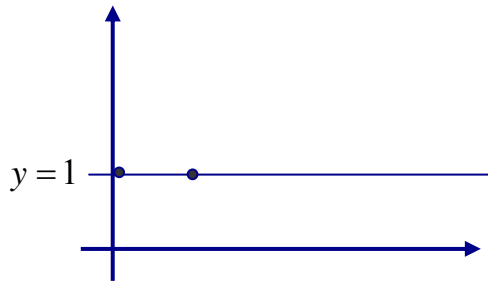
The line through two points x and x' is the line $l = x \times x' = [x]_x x' = -[x']_x x$



Lines from Points - Line joining two Points

The line through two points x and x' is the line $l = x \times x' = [x]_x x' = -[x']_x x$

Example



$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Ideal Points and the Line at Infinity

Intersections of parallel lines are the points at infinity (ideal points)

$$l = (a, b, c)^T \text{ and } l' = (a, b, c')^T \quad l \times l' = (b, -a, 0)^T$$

$$[l]_x l' = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c' \end{bmatrix} = \begin{bmatrix} b(c'-c) \\ a(c-c') \\ 0 \end{bmatrix} \approx \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$$

Ideal points $(x_1, x_2, 0)^T$

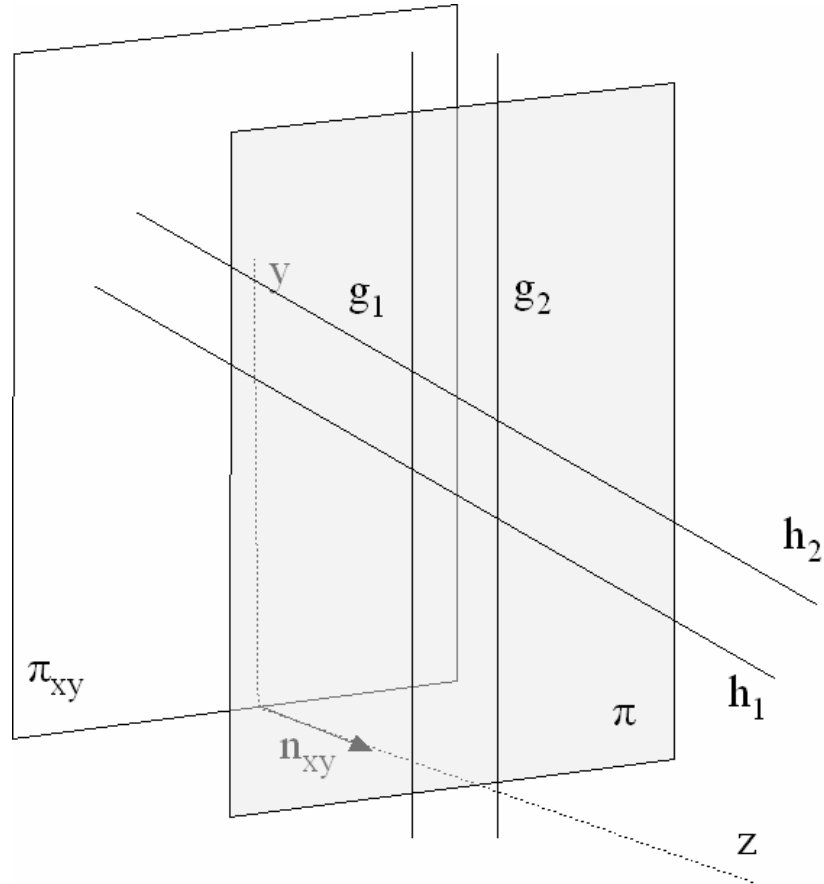
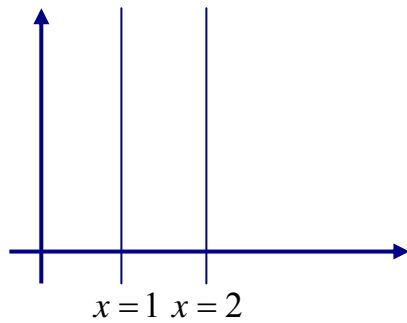
Line at infinity $l_\infty = (0, 0, 1)^T$

alternative definition of \mathbf{P}^2 : $\mathbf{P}^2 = \mathbf{R}^2 \cup l_\infty$

Note that in \mathbf{P}^2 there is no distinction between ideal points and others

Ideal points and the line at infinity – an example

Example



Duality

$$\begin{array}{ccc} \mathbf{x} & \longleftrightarrow & l \\ \mathbf{x}^\top \mathbf{l} = 0 & \longleftrightarrow & l^\top \mathbf{x} = 0 \\ \mathbf{x} = l \times l' & \longleftrightarrow & l = \mathbf{x} \times \mathbf{x}' \end{array}$$

Duality principle:

To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the role of points and lines in the original theorem

Summary

- Projective Space P^2
- Homogeneous Coordinates
- Points
- Lines
- Points and Lines at the Infinity
- Intersection of Lines: Points
- Connection of Points: Lines
- Duality

Readings

- For this lecture:
Hartley & Zisserman pp.:
 - 25-32 (P^2 Projective Plane)

- For the next lecture:
Hartley & Zisserman pp.:
 - 32-49 (P^2 Projective Transformations),
 - 65-73 (3D Projective Space)