

# 3D Computer Vision Transformations & 2D & 3D Projective Geometry

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based on a course given at UNC by Marc Pollefeys & the book "Multiple View Geometry" by Hartley & Zisserman

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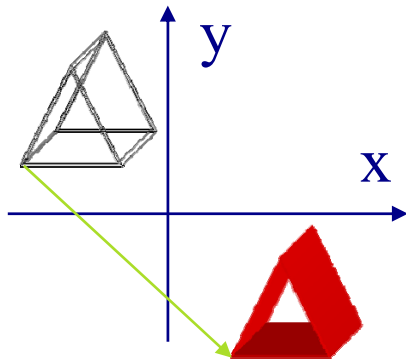
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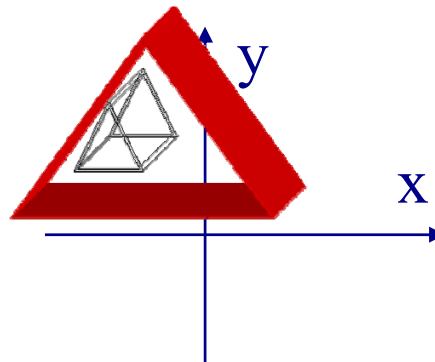
# 3D Transformations

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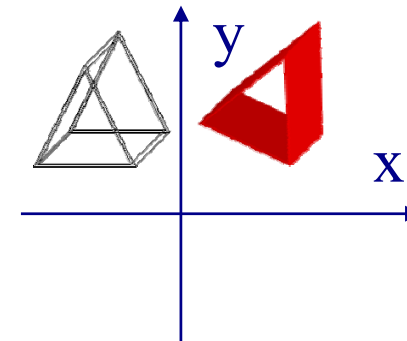
## Basic Coordinate Transformations



translation



scaling



rotation

## 3D Transformations - Translation

3D Point

$$\vec{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Translation

$$\vec{p}' = \vec{p} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

## 3D Transformations - Translation

Translation Matrix

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I & T \\ 0 & 1 \end{bmatrix}$$

Translation

$$\vec{p}' = \begin{bmatrix} I & T \\ 0 & 1 \end{bmatrix} \vec{p}$$

## 3D Transformations - Rotation

Rotation Matrix

$$\begin{bmatrix} & & & 0 \\ & R & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R & \vec{0} \\ \vec{0} & 1 \end{bmatrix}$$

Rotation

$$\vec{p} \rightarrow \begin{bmatrix} R & \vec{0} \\ \vec{0} & 1 \end{bmatrix} \vec{p}$$

## 3D Transformations - Rotation

- Euler Angles for Rotation  $R=R_zR_yR_x$  :

rotation by  $\psi$  about the z axis:  $R_z = \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

rotation by  $\theta$  about the y axis:  $R_y = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$

rotation by  $\phi$  about the x axis:  $R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{bmatrix}$

## 3D Transformations - Euclidean Transformation

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

## 3D Transformations - Metric Transformation

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \\ 0 & 0 & 0 \end{bmatrix} T \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

## 3D Transformations - Affine Transformations

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \\ \vec{0} \end{bmatrix} \cdot \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \\ 1 \end{bmatrix} T \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Non Isotropic Scaling

## 3D Transformations - Affine Transformations

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} \alpha_u & 0 & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \\ \vec{0} & & & 1 \end{bmatrix} \cdot \begin{bmatrix} R & T \\ & & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Usual Camera Model: image center & aspect ratios

## 3D Transformations - Affine Transformations

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{21} & q_{22} & q_{23} & q_{24} \\ q_{31} & q_{32} & q_{33} & q_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

General Affine Transform

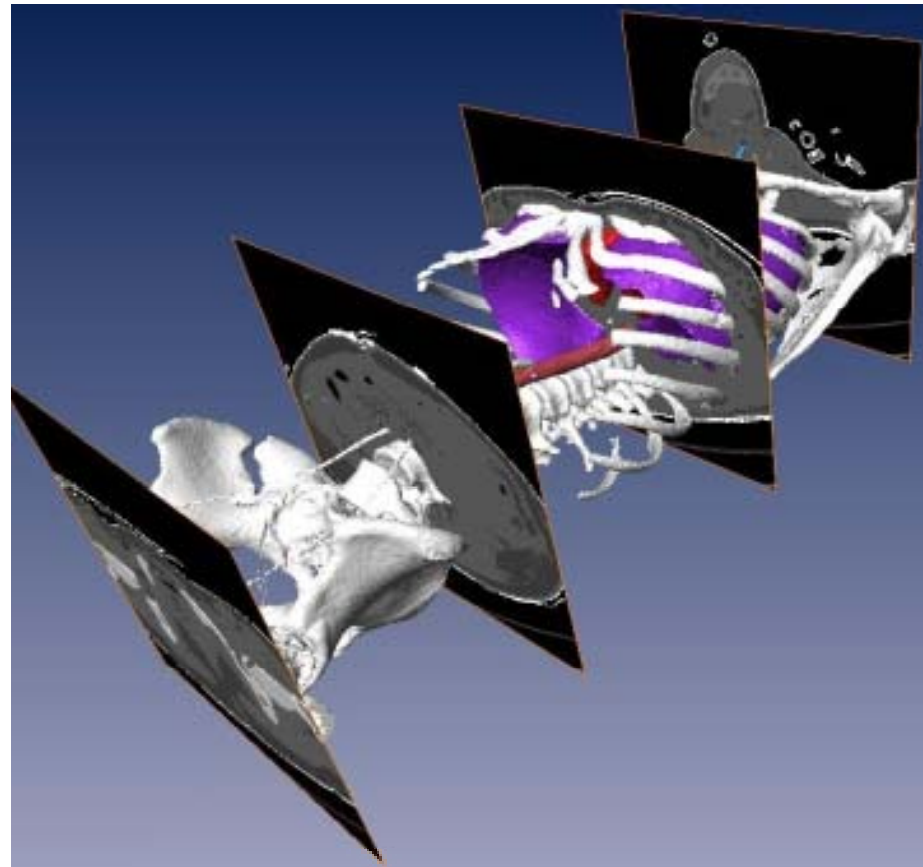
## 3D Transformations - Affine Transformations

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \\ \vec{0} & & 1 \end{bmatrix} \cdot \begin{bmatrix} R & T \\ & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

General Affine Transformation

# 3D Projective Transformations

Transformations of  $P^3$  are very similar to the transformations of  $P^2$



## 3D Projective Transformations - Projective Transformations

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

General Projective Transformation

$$R^t R = R^{-1} R = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_i^t R_i = 1$$

$$R_i^t R_j = \emptyset$$

$$1 \times e^{i\theta} \times e^{-i\theta}$$

$$1, \cos \theta + i \sin \theta, \cos \theta - i \sin \theta$$

$$R X = X$$

$$R \vec{v} = \vec{v}$$

$$\arccos\left(\frac{\text{Trace}(R) - 1}{2}\right)$$

# 3D Projective Transformations - Projective Transformations

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \\ p_{41} & p_{42} & p_{43} \end{bmatrix} \cdot \begin{bmatrix} [R & T] \\ 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

General Projective Transformation

## 3D Projective Space - 3D Points

3D point  $(X, Y, Z)^T$  in  $\mathbf{R}^3$

$$\mathbf{X} = (X_1, X_2, X_3, X_4)^T \text{ in } \mathbf{P}^3$$

$$\mathbf{X} = \left( \frac{X_1}{X_4}, \frac{X_2}{X_4}, \frac{X_3}{X_4}, 1 \right)^T = (X, Y, Z, 1)^T \quad (X_4 \neq 0)$$

projective transformation

$$\mathbf{X}' = \mathbf{H} \mathbf{X} \quad (4 \times 4 - 1 = 15 \text{ dof})$$

## 3D Projective Space - Planes

3D plane  $\pi_1 X + \pi_2 Y + \pi_3 Z + \pi_4 = 0$

$$\pi_1 X_1 + \pi_2 X_2 + \pi_3 X_3 + \pi_4 X_4 = 0$$

$$\pi^T \mathbf{X} = 0$$

Transformation of a plane by point homography  $\mathbf{H}$

$$\begin{aligned} \mathbf{X}' &= \mathbf{H} \mathbf{X} \\ \pi' &= \mathbf{H}^{-T} \pi \end{aligned}$$

Duality: points  $\leftrightarrow$  planes, lines  $\leftrightarrow$  lines

# 3D Projective Space - Planes

a reminder: Euclidean representation

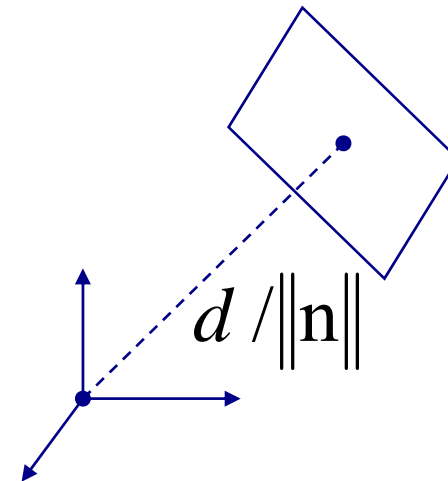
$$\mathbf{n} \cdot \tilde{\mathbf{X}} + d = 0$$

$$\mathbf{n} = (\pi_1, \pi_2, \pi_3)^\top$$

$$\pi_4 = d$$

$$\tilde{\mathbf{X}} = (X, Y, Z)^\top$$

$$X_4 = 1$$



## 3D Projective Space - Planes from Points

Solve  $\pi$  from  $X_1^\top \pi = 0$ ,  $X_2^\top \pi = 0$  and  $X_3^\top \pi = 0$

$$\begin{bmatrix} X_1^\top \\ X_2^\top \\ X_3^\top \end{bmatrix} \pi = 0 \quad (\text{solve } \pi \text{ as right nullspace of } \begin{bmatrix} X_1^\top \\ X_2^\top \\ X_3^\top \end{bmatrix} )$$

## 3D Projective Space - Planes from Points II

Implicitly from coplanarity condition  $\det[X X_1 X_2 X_3] = 0$

$$\det \begin{bmatrix} X_1 & (X_1)_1 & (X_2)_1 & (X_3)_1 \\ X_2 & (X_1)_2 & (X_2)_2 & (X_3)_2 \\ X_3 & (X_1)_3 & (X_2)_3 & (X_3)_3 \\ X_4 & (X_1)_4 & (X_2)_4 & (X_3)_4 \end{bmatrix} = 0$$

$$X_1 D_{234} - X_2 D_{134} + X_3 D_{124} - X_4 D_{123} = 0$$

$$\pi = (D_{234}, -D_{134}, D_{124}, -D_{123})^T$$

## 3D Projective Space - Planes from Points II

$$\pi = (D_{234}, -D_{134}, D_{124}, -D_{123})^T$$

If  $\mathbf{X}_i = (\tilde{\mathbf{X}}_i, 1)^T$  and  $\tilde{\mathbf{X}}_i = (X_i, Y_i, Z_i)$ , we have :

$$D_{234} = \det \begin{bmatrix} Y_1 & Y_2 & Y_3 \\ Z_1 & Z_2 & Z_3 \\ 1 & 1 & 1 \end{bmatrix} = \det \begin{bmatrix} Y_1 - Y_3 & Y_2 - Y_3 & Y_3 \\ Z_1 - Z_3 & Z_2 - Z_3 & Z_3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= ((\tilde{\mathbf{X}}_1 - \tilde{\mathbf{X}}_3) \times (\tilde{\mathbf{X}}_2 - \tilde{\mathbf{X}}_3))_1$$

$$\pi = \begin{bmatrix} ((\tilde{\mathbf{X}}_1 - \tilde{\mathbf{X}}_3) \times (\tilde{\mathbf{X}}_2 - \tilde{\mathbf{X}}_3)) \\ -\tilde{\mathbf{X}}_3^t (\tilde{\mathbf{X}}_1 \times \tilde{\mathbf{X}}_2) \end{bmatrix}$$

## 3D Projective Space - Points from Planes

Solve  $X$  from  $\pi_1^\top X = 0$ ,  $\pi_2^\top X = 0$  and  $\pi_3^\top X = 0$

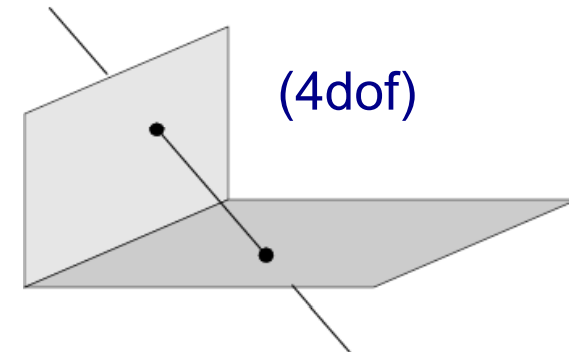
$$\begin{bmatrix} \pi_1^\top \\ \pi_2^\top \\ \pi_3^\top \end{bmatrix} X = 0 \quad (\text{solve } X \text{ as right nullspace of } \begin{bmatrix} \pi_1^\top \\ \pi_2^\top \\ \pi_3^\top \end{bmatrix})$$

Representing a plane by its span

$$\begin{aligned} X &= \mathbf{M} x & \mathbf{M} &= [X_1 X_2 X_3] & \pi &= (a, b, c, d)^\top \\ \pi^\top \mathbf{M} &= 0 & \mathbf{M} &= \begin{bmatrix} p \\ \mathbf{I} \end{bmatrix} & p &= \left( -\frac{b}{a}, -\frac{c}{a}, -\frac{d}{a} \right)^\top \end{aligned}$$

## 3D Projective Space - Lines

- Line representation in 3D is awkward:
  - lines in 3D have 4 DOF
  - thus we would have to represent them by a 5 element homogeneous vector
  - we need other representations
  
- Presented representations in the lesson:
  - Span-Representation
  - Plücker Matrices
  - Plücker Line Coordinates



## 3D Projective Space – Line Representation

- Null-Space and Span Representation
  - a line in 3D can be represented as a span of a matrix  $W$  consisting of two points  $A$  and  $B$ , which are lying on the line

$$W = \begin{bmatrix} A^T \\ B^T \end{bmatrix} \quad \lambda A + \mu B$$

- the dual representation as the right-null space of the matrix  $W^*$  containing two planes  $P$  and  $Q$  which intersect in the line

$$W^* = \begin{bmatrix} P^T \\ Q^T \end{bmatrix} \quad \lambda P + \mu Q$$

- the two representations are related as follows:  $W^* W^T = W W^{*T} = 0_{2 \times 2}$

## 3D Projective Space – Lines Representation

- Null-Space and Span Representation

Example: the X-axis

X-axis as span of the points  
(0,0,0) and the X-axis-direction (1,0,0)

$$W = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

X-axis as right null-space of the matrix of  
the XY-plane (0,0,1) and the XZ-plane

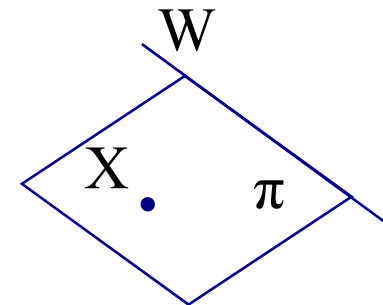
$$W^* = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

# 3D Projective Space

## Points, Lines and Planes – Joins and Intersections

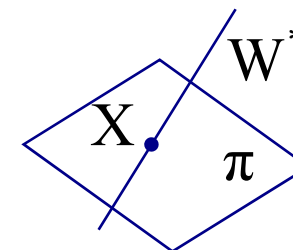
- the plane  $\pi$  which is the join of the line  $W$  and the point  $X$  is the right null-space of the matrix  $M$  containing  $W$  and  $X$ .

$$\mathbf{M} = \begin{bmatrix} W \\ X^\top \end{bmatrix} \quad \mathbf{M} \pi = 0$$



- the point  $X$  that is the intersection point of the plane  $\pi$  and the line  $W^*$  is the right null-space of the matrix  $M$  consisting of  $M W^*$  and  $\pi$ .

$$\mathbf{M} = \begin{bmatrix} W^* \\ \pi^\top \end{bmatrix} \quad \mathbf{M} X = 0$$



# 3D Projective Space – Line Representation

- Plücker Matrices
  - Plücker matrix (4x4 skew-symmetric homogeneous matrix)
  - For the line joining the points A and B, the Plücker Matrix L is:

$$l_{ij} = A_i B_j - B_i A_j$$

$$L = AB^T - BA^T$$

## 3D Projective Space – Line Representation

- Plücker Matrices

- Properties of Plücker Matrices:

1.  $L$  has rank 2:  $LW^{*\top} = 0_{4 \times 2}$
2. 4DOF
3. generalization of  $\mathbf{l} = \mathbf{x} \times \mathbf{y}$
4.  $L$  independent of choice of the line points  $A$  and  $B$
5. Transformation of  $L$ :  $L' = HLH^\top$

# 3D Projective Space – Line Representation

- Plücker Matrices

Example: the X-axis can be represented as

$$L = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

## 3D Projective Space – Line Representation

### □ Plücker Matrices

Dual Plücker matrix  $L^*$  for the intersection line of the planes P and Q

$$L^* = PQ^T - QP^T$$

Transformation of the dual Plücker matrix  $L^*$

$$L^{*'} = H^{-T} L H^{-1}$$

Correspondence of the Plücker Matrix L and its dual  $L^*$

$$l_{12} : l_{13} : l_{14} : l_{23} : l_{42} : l_{34} = l_{34}^* : l_{42}^* : l_{23}^* : l_{14}^* : l_{13}^* : l_{12}^*$$

# 3D Projective Space – Line Representation

## □ Plücker Matrices

Join and incidence

$$\pi = L^* X \quad (\text{plane through point and line})$$

$$L^* X = 0 \quad (\text{point on line})$$

$$X = L\pi \quad (\text{intersection point of plane and line})$$

$$L\pi = 0 \quad (\text{line in plane})$$

$$[L_1, L_2, \dots]\pi = 0 \quad (\text{coplanar lines})$$

## 3D Projective Space – Line Representation

- Plücker Line Coordinates
  - The Plücker Line Coordinates  $L$  are the six non-zero elements of the Plücker matrix

$$L = [l_{12}, l_{13}, l_{14}, l_{23}, l_{42}, l_{34}] \in \mathbf{P}^5$$

- This can be interpreted as a homogeneous 6-vector, thus an element of 5D Projective Space
- It can be verified that the following equation holds

$$l_{12}l_{34} + l_{13}l_{42} + l_{14}l_{23} = 0$$

## 3D Projective Space – Line Representation

- Plücker Line Coordinates
  - It can be verified that the following equation holds:

$$l_{12}l_{34} + l_{13}l_{42} + l_{14}l_{23} = 0$$

- It can be shown that a 6-vector only corresponds to a line in 3D-space if the above equation holds.

## 3D Projective Space – Line Representation

- Plücker Line Coordinates
  - coplanar lines:
    - two lines through points A,B and  $\hat{A}, \hat{B}$  are coplanar if

$$\det[A, B, \hat{A}, \hat{B}] = 0$$

- it can be shown that the formula expands like:

$$\det[A, B, \hat{A}, \hat{B}] = l_{12}\hat{l}_{34} + l_{13}\hat{l}_{42} + l_{14}\hat{l}_{23} + l_{23}\hat{l}_{14} + l_{42}\hat{l}_{13} + l_{34}\hat{l}_{12} = (\mathbf{L} | \hat{\mathbf{L}})$$

- this product helps expressing the following relationships...

# 3D Projective Space – Line Representation

- Plücker Line Coordinates

- the derived formula can be written as a matrix-vector-product

$$(L | \hat{L}) = \det[A, B, \hat{A}, \hat{B}] = l_{12}\hat{l}_{34} + l_{13}\hat{l}_{42} + l_{14}\hat{l}_{23} + l_{23}\hat{l}_{14} + l_{42}\hat{l}_{13} + l_{34}\hat{l}_{12}$$

$$= \begin{bmatrix} l_{12} & l_{13} & l_{14} & l_{23} & l_{42} & l_{34} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{l}_{12} \\ \hat{l}_{13} \\ \hat{l}_{14} \\ \hat{l}_{23} \\ \hat{l}_{42} \\ \hat{l}_{34} \end{bmatrix}$$

- $(L | \hat{L})$  will help us express the following relationships...

## 3D Projective Space – Line Representation

- Plücker Line Coordinates

$$(\mathbf{L} | \mathbf{L}) = 0 \quad (\text{Plücker internal constraint})$$

$$(\mathbf{L} | \hat{\mathbf{L}}) = \det[\mathbf{A}, \mathbf{B}, \hat{\mathbf{A}}, \hat{\mathbf{B}}] = 0 \quad (\text{two lines intersect})$$

$$(\mathbf{L} | \hat{\mathbf{L}}) = \det[\mathbf{P}, \mathbf{Q}, \hat{\mathbf{P}}, \hat{\mathbf{Q}}] = 0 \quad (\text{two lines intersect})$$

$$(\mathbf{L} | \hat{\mathbf{L}}) = (\mathbf{P}^\top \mathbf{A})(\mathbf{Q}^\top \mathbf{B}) - (\mathbf{Q}^\top \mathbf{A})(\mathbf{P}^\top \mathbf{B}) = 0 \quad (\text{two lines intersect})$$

## 3D Projective Space - The Plane at Infinity

$$\pi'_\infty = \mathbf{H}_A^{-T} \pi_\infty = \begin{bmatrix} \mathbf{A}^{-T} & 0 \\ -(\mathbf{A} \mathbf{t})^T & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \pi_\infty$$

The plane at infinity  $\pi_\infty$  is a fixed plane under a projective transformation  $H$  iff  $H$  is an affinity

1. canonical position  $\pi_\infty = (0,0,0,1)^T$
2. contains directions  $D = (X_1, X_2, X_3, 0)^T$
3. two planes are parallel  $\Leftrightarrow$  line of intersection in  $\pi_\infty$
4. line // line (or plane)  $\Leftrightarrow$  point of intersection in  $\pi_\infty$

# Summary

- Projective Transformations (2D)
- Transformations of Points and Lines
- Hierarchy of Transformations
  - Euclidean
  - Similarity
  - Affine
  - Projective
- 3D Projective Space
  - Points
  - Planes
  - Lines

## Readings for the next Lecture

- Hartley & Zisserman pp.
  - 30-32 (Conics)
  - 47-64 (Recovery of Properties from Images - Conics)
  - 73-85 (Quadrics and Properties)