

Exercises in 3D Computer Vision

Exercise 18 **Conics**

Conics can be described using quadratic forms. A point \mathbf{p} on a conic verifies: $\mathbf{p}^\top \mathbf{C} \mathbf{p} = 0$, where \mathbf{C} is the matrix of the quadratic form, which is symmetric. Under the projective transformation \mathbf{H} , a conic C having the quadratic form matrix \mathbf{C} is transformed to a conic C' having the quadratic form matrix \mathbf{C}' with $\mathbf{C}' \propto \mathbf{H}^{-\top} \mathbf{C} \mathbf{H}^{-1}$.

- a) Verify this transformation property.
- b) Show that a conic C having the quadratic form matrix

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ -1 & -2 & -4 \end{bmatrix}$$

is a circle. Give the coordinates of its center and give its radius.

- c) Show that a conic C having the quadratic form matrix \mathbf{C} and two points, \mathbf{p}_1 and \mathbf{p}_2 , in general position have the invariant

$$Inv = \frac{(\mathbf{p}_1^\top \mathbf{C} \mathbf{p}_2)^2}{(\mathbf{p}_1^\top \mathbf{C} \mathbf{p}_1)(\mathbf{p}_2^\top \mathbf{C} \mathbf{p}_2)}.$$

- d) A similarity transformation leaves two points invariant - the circular points $\mathbf{a} = [1, i, 0]^\top$ and $\mathbf{b} = [1, -i, 0]^\top$. Show it.
- e) Proof that the conic dual to the circular points, $\mathbf{C}_\infty^* = \mathbf{a} \mathbf{b}^\top + \mathbf{b} \mathbf{a}^\top$ is also invariant under a similarity transformation.
- f) If two lines \mathbf{l} and \mathbf{m} are given in \mathbb{R}^2 , how can the angle between the two be determined? Develop a formula that determines the angle between the two lines in \mathbb{P}^2 and is invariant to projective transformations.

Exercise 19 **Non-linear estimation methods**

The problem of estimating a homography using sets of corresponding points $(\bar{\mathbf{x}}_i, \bar{\mathbf{x}}'_i)$, $i \in \{1, \dots, n\}$ can be solved non-linearly. This is necessary for minimizing geometric cost functions. Non-linear optimizations use iterative algorithms such as the **Gradient Descent**, **Gauss-Newton** or **Levenberg-Marquardt** algorithms. The common starting point is a function f that depends on a parameter vector $\mathbf{p} \in \mathbb{R}^M$. Here, f is not linear with respect to \mathbf{p} .

- a) Determining the first derivative (Jacobian) of the function f with respect to \mathbf{p} is necessary. Develop the Jacobian for the problem of estimating a homography out of point correspondences assuming errors only in one image. (Hint: the parameter vector \mathbf{p} consists only in the entries of \mathbf{H}).

- b) Develop the Jacobian for homography estimation using the reprojection error (Hint: the parameter vector \mathbf{p} consists in the entries of \mathbf{H} and the modified coordinates $\{\mathbf{x}_i\}$ of the measurement points $\{\bar{\mathbf{x}}_i\}$).
- c) **(H)** Program the “Gold Standard Algorithm” for estimating a homography in MatLab. The algorithm is given below.
- (i) Compute an initial estimate of $\hat{\mathbf{H}}$ to provide a starting point for the geometric minimization. For instance, use the normalized DLT algorithm.
 - (ii) Use an initial estimate of the missing entries $\{\mathbf{x}_i\}$ of \mathbf{p} , e.g. use the measurement points $\{\bar{\mathbf{x}}_i\}$.
 - (iii) Minimize the cost

$$\sum_i (d(\bar{\mathbf{x}}_i, \mathbf{x}_i)^2 + d(\bar{\mathbf{x}}'_i, \mathbf{x}'_i)^2), \text{ where } \mathbf{x}'_i \propto \mathbf{H}\mathbf{x}_i$$

using the Levenberg-Marquardt algorithm over the $9 + 2n$ variables in \mathbf{p} .

The function that estimates the homography with the Gold Standard Algorithm should be delivered with a Matlab script. The Matlab script

- defines a reference homography
- creates more than 50 pairs of corresponding points in 2 images (all the points should fit into images of size 640×480)
- adds Gaussian noise to the points' coordinates
- displays the created noisy points in the 2 images
- solves the homography using the Gold Standard Algorithm and the the created corresponding points (the noisy ones)
- compares the two homographies: the reference homography and the homography estimated with the Gold Standard Algorithm.