

Augmented Reality II

- Projective Geometry -

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April 20, 2004 and April 27, 2004

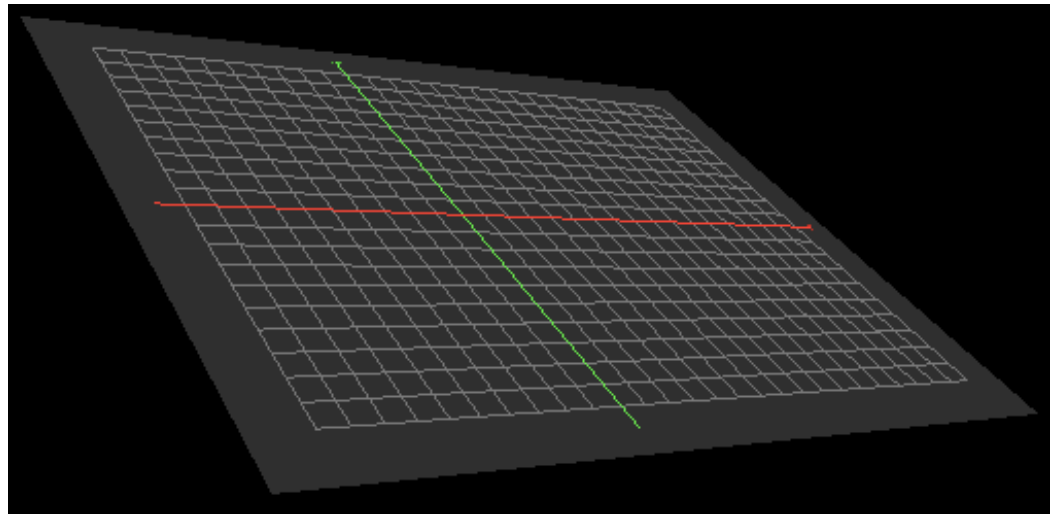
Part 1:
Projective Geometry and
Transformations in 2D

Literature

- Richard Hartley and Andrew Zisserman, “Multiple View Geometry in Computer Vision”, Cambridge University Press, 2000. (Section 1)
- Illustrations (VRML) at [illustrations/ProjGeom2D/](#)

Projective Geometry and Transformations in 2D

- Geometric distortion due to perspective Projection
- *Invariant:*
 - Straight lines
- **Not invariant:**
 - Angles
 - Parallel lines

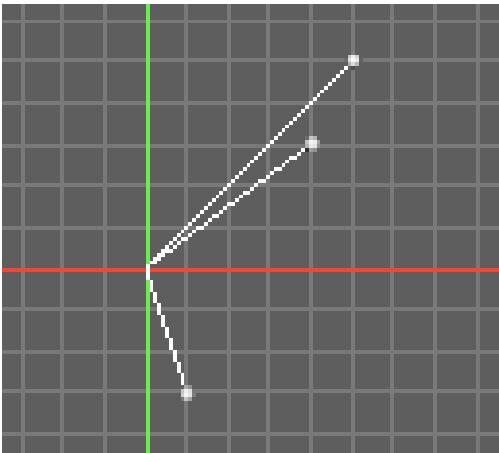


2D Projective Plane

- Points -

- *Inhomogeneous* notation in \mathbb{R}^2

$$P = (x, y)^T$$



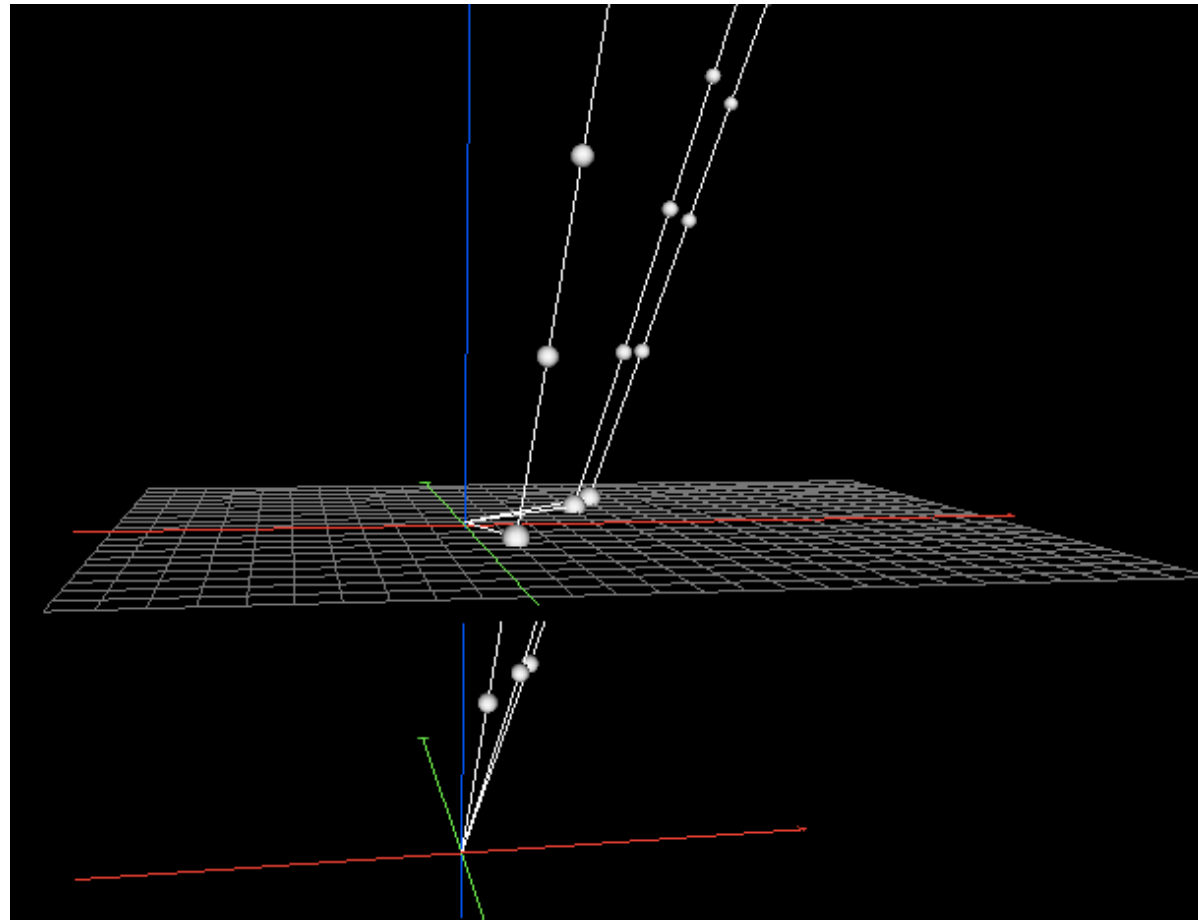
- *Homogeneous* notation in \mathbb{P}^2
(*projective space*)

$$\mathbf{x} = (wx, wy, w)^T$$

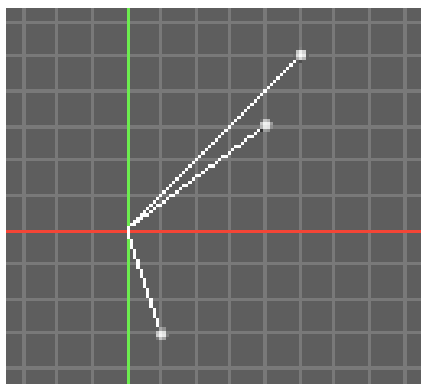
$$= (x, y, 1)^T, \text{ with } w = 1$$

2D Projective Plane

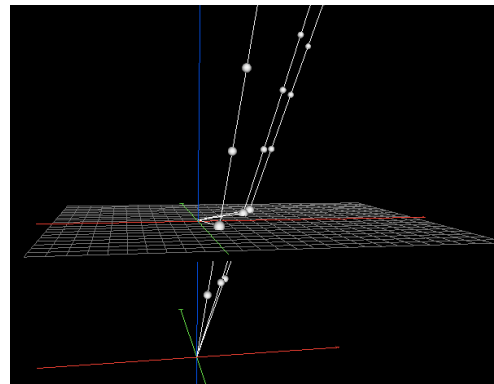
- Points -



$$\mathbf{x} = w(x, y, 1)^T$$


 \mathbb{R}^2

Examples


 \mathbb{P}^2

$$P_1 = (0.4, 0.3)^T$$

$$\mathbf{x}_1 = (0.4w, 0.3w, w)^T$$

$$= (0.4, 0.3, 1.0)^T$$

$$= (0.8, 0.6, 2.0)^T$$

$$P_2 = (0.1, -0.3)^T$$

$$\mathbf{x}_2 = (0.3, -0.9, 3.0)^T$$

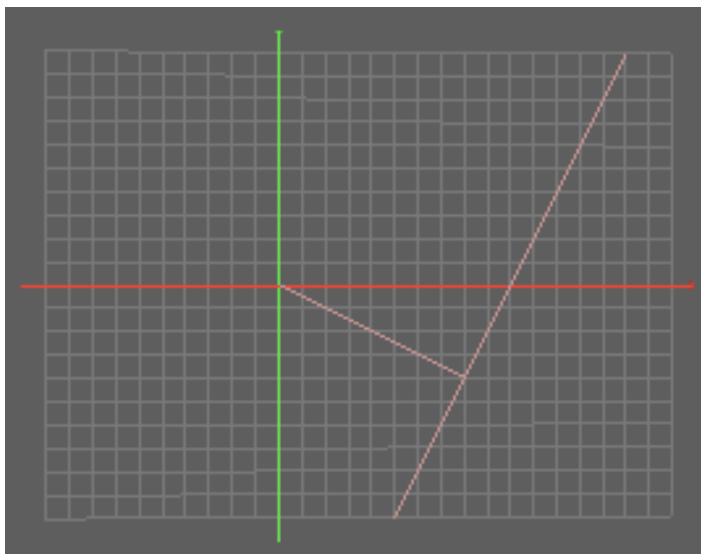
$$P_3 = (0.5, 0.5)^T$$

$$\mathbf{x}_3 = (0.5, 0.5, 1.0)^T$$

2D Projective Plane

- Lines -

- Line equation: $ax + by + c = 0$
 Line normal: $\mathbf{n} = (a, b)/|\mathbf{n}|$
- Homogeneous line notation
 in P^2 (projective space): $\mathbf{l} = k(a, b, c)^T$

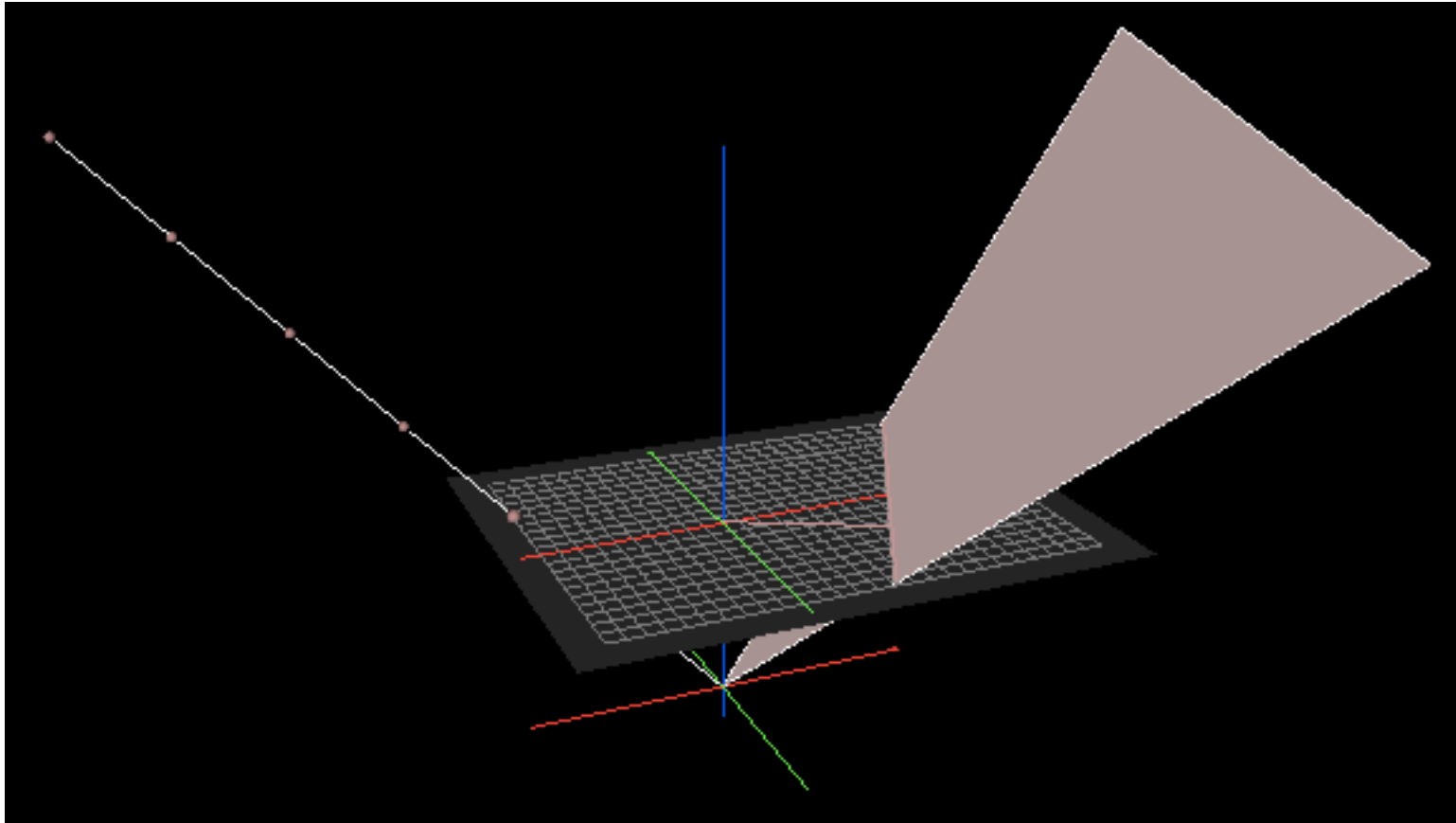


$$w(x, y, 1) \cdot k \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

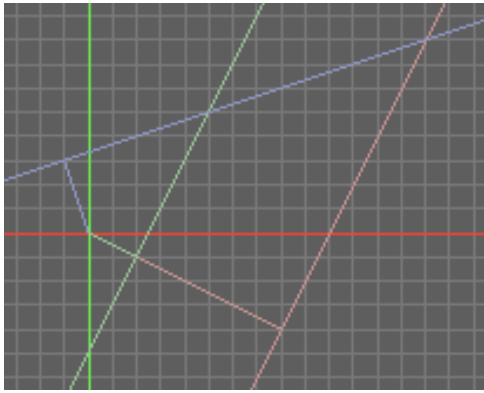
$$\mathbf{x}^T \mathbf{l} = \mathbf{l}^T \mathbf{x} = 0$$

2D Projective Plane

- Lines -

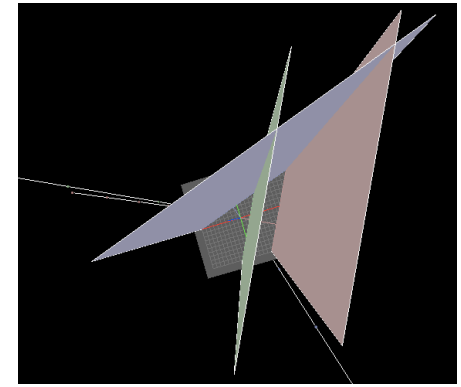


$$\mathbf{l} = k(a, b, c)^T$$



\mathbb{R}^2

Examples



\mathbb{P}^2

$$l_1 : 2x - y - 2 = 0$$

$$\mathbf{l}_1 = (2, -1, -2)^T$$

$$= (-1, 0.5, 1.0)^T$$

$$l_2 : 2x - y - 0.5 = 0$$

$$\mathbf{l}_2 = (2, -1, -0.5)^T$$

$$= (-4, 2, 1)^T$$

$$l_3 : x - 3y + 1 = 0$$

$$\mathbf{l}_3 = (1, -3, 1)^T$$

2D Projective Plane

- Comparison -

\mathbb{R}^2

\mathbb{P}^2

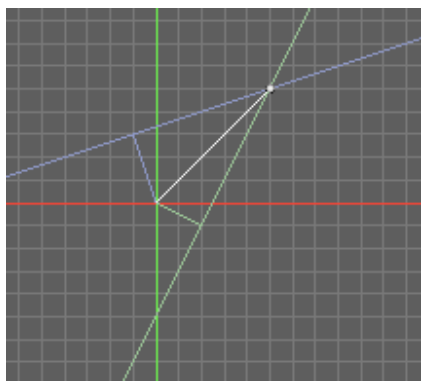
- Degrees of Freedom (*DOF*): 2
 - Point $P = (x,y)$
 - Line $l: ax+by+c=0$
Normal $\mathbf{n}=(a,b)/|\mathbf{n}|$
- Degrees of Freedom (DOF): 2
 - Vector $\mathbf{x} = w(x,y,1)^T$
 - Vector $\mathbf{l} = k(a,b,c)^T$
 - *Duality*: $\mathbf{x}^T \mathbf{l} = \mathbf{l}^T \mathbf{x} = 0$

2D Projective Plane

- Intersection of Lines -

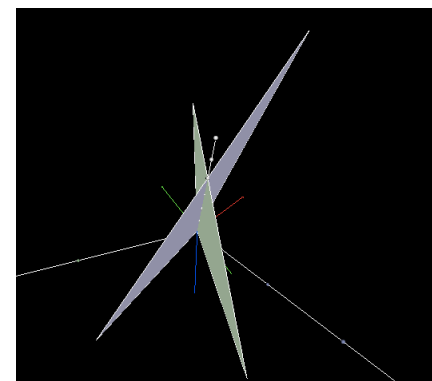
- Lines $\mathbf{l}_1 = (a_1, b_1, c_1)^T$ and $\mathbf{l}_2 = (a_2, b_2, c_2)^T$ intersect at a point $\mathbf{x} = (x, y, w)^T$.
- \mathbf{x} is on \mathbf{l}_1 and on \mathbf{l}_2 : $\mathbf{x}^T \mathbf{l}_1 = 0$, $\mathbf{x}^T \mathbf{l}_2 = 0$
- \mathbf{x} is perpendicular to \mathbf{l}_1 and \mathbf{l}_2 .
- \mathbf{x} is cross product of \mathbf{l}_1 and \mathbf{l}_2 .

$$\mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2 = \begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{bmatrix} b_1 c_2 - c_1 b_2 \\ c_1 a_2 - a_1 c_2 \\ a_1 b_2 - b_1 a_2 \end{bmatrix}$$



\mathbb{R}^2

Example



\mathbb{P}^2

$$l_2 : 2x - y - 0.5 = 0$$

$$l_3 : x - 3y + 1 = 0$$

$$P_3 = (0.5 \quad 0.5)^T$$

$$\mathbf{l}_2 = (-4.0, \quad 2.0, \quad 1.0)^T$$

$$\mathbf{l}_3 = (1.0, \quad -3.0, \quad 1.0)^T$$

$$\mathbf{x}_3 = (0.5, \quad 0.5, \quad 1.0)^T$$

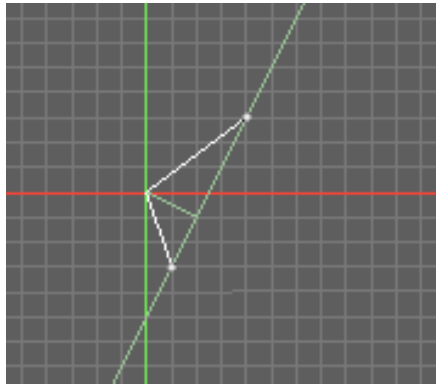
$$\begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \\ 10 \end{bmatrix}$$

2D Projective Plane

- Line through 2 Points -

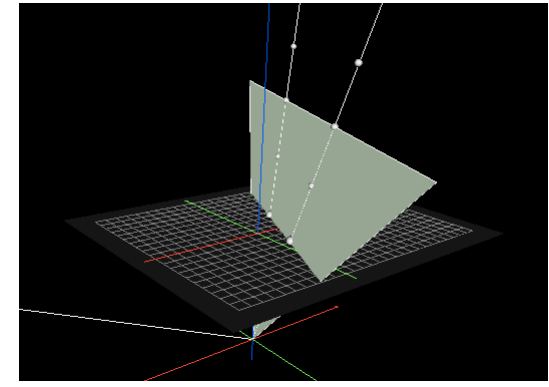
- Points $\mathbf{x}_1 = (x_1, y_1, w_1)^T$ and $\mathbf{x}_2 = (x_2, y_2, w_2)^T$ define a line $\mathbf{l} = (a, b, c)^T$.
- \mathbf{l} goes through \mathbf{x}_1 and \mathbf{x}_2 : $\mathbf{x}_1^T \mathbf{l} = 0$, $\mathbf{x}_2^T \mathbf{l} = 0$
- \mathbf{l} is perpendicular to both vectors.
- \mathbf{l} is cross product of \mathbf{x}_1 and \mathbf{x}_2 .

$$\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2$$



\mathbb{R}^2

Example



\mathbb{P}^2

$$P_1 = (0.4, 0.3)^T$$

$$P_2 = (0.1, -0.3)^T$$

$$l_2 : 2x - y - 0.5 = 0$$

$$\mathbf{x}_1 = (0.4, 0.3, 1.0)^T$$

$$\mathbf{x}_2 = (0.1, -0.3, 1.0)^T$$

$$\mathbf{l}_2 = (2.0, -1.0, -0.5)^T$$

$$\begin{bmatrix} 0.4 \\ 0.3 \\ 1.0 \end{bmatrix} - \begin{bmatrix} 0.1 \\ -0.3 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.6 \\ -0.15 \end{bmatrix} = 6.66 \begin{bmatrix} 0.04 \\ 0.02 \\ -0.01 \end{bmatrix}$$

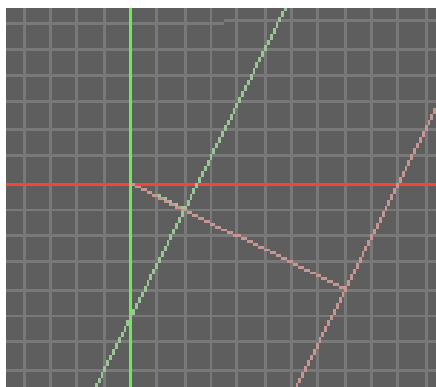
2D Projective Plane

- Ideal Points -

- Intersection of parallel lines $\mathbf{l}_1 = (a,b,c)^T$ and $\mathbf{l}_2 = (a,b,c')^T$

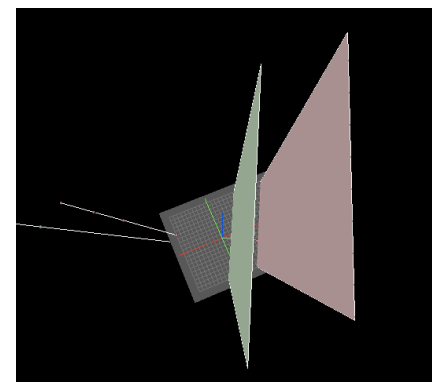
$$\mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2 = (c' - c) \begin{bmatrix} b \\ a \\ 0 \end{bmatrix}$$

- Parallel lines intersect “at infinity”.
- *Ideal points* lie on plane $w=0$
(*Points at infinity*).



\mathbb{R}^2

Example



\mathbb{P}^2

$$l_1 : 2x - y - 2 = 0$$

$$l_2 : 2x - y - 0.5 = 0$$

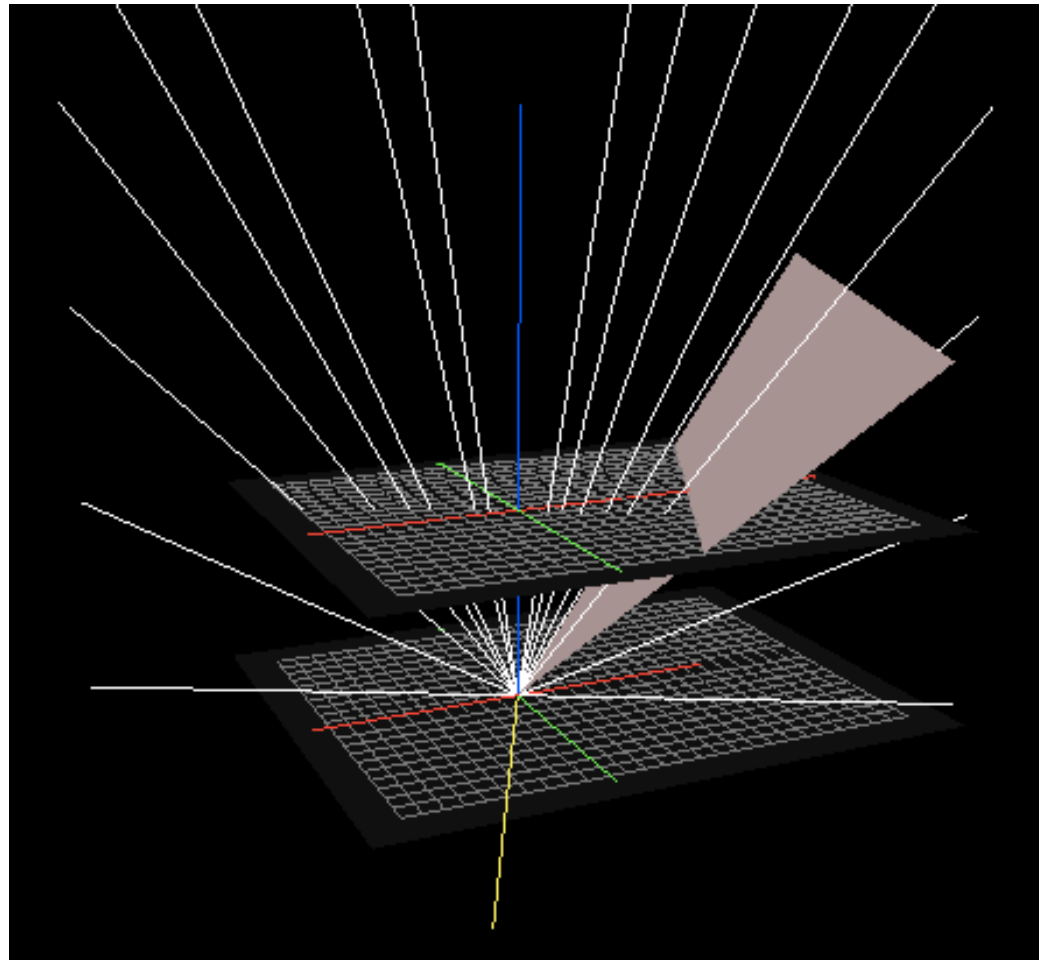
$$\mathbf{l}_2 = (2, -1, -2)^T$$

$$\mathbf{l}_3 = (2, -1, -0.5)^T$$

$$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1.5 \end{bmatrix} = 1.5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

2D Projective Plane

- Ideal Points -



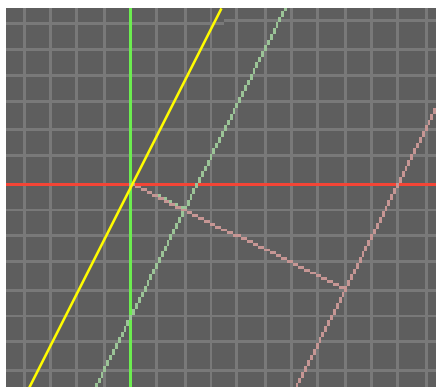
2D Projective Plane

- Points at Infinity -

- Set of all ideal points (*points at infinity*):

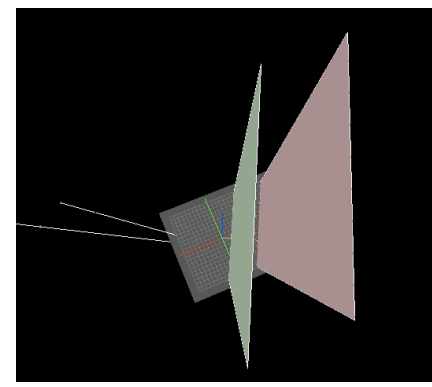
$$\begin{aligned}\mathbf{x}_{\text{Id}_i} &= (x_i, y_i, 0)^T \\ &= s(x_i/y_i, 1, 0)^T\end{aligned}$$

i.e.: all ideal points lie in plane, $w = 0$.



\mathbb{R}^2

Example



\mathbb{P}^2

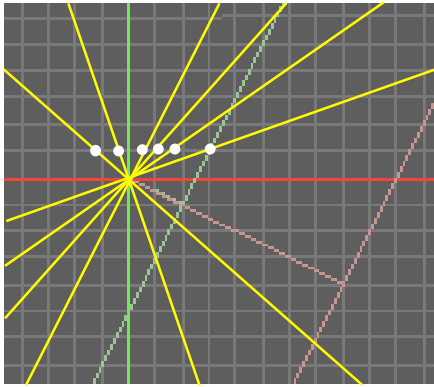
$$l_1 : 2x - y - 2 = 0$$

$$l_2 : 2x - y - 0.5 = 0$$

$$\mathbf{l}_2 = (2, -1, -2)^T$$

$$\mathbf{l}_3 = (2, -1, -0.5)^T$$

$$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1.5 \end{bmatrix} = 1.5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



2D Projective Plane

- Line at Infinity -

- Set of all ideal points (*points at infinity*):
 $\mathbf{x}_{Id_i} = (x_i, y_i, 0)^T = s(x_i/y_i, 1, 0)^T$
 i.e.: all ideal points lie in plane, $w = 0$.
- The *line at infinity* represents all ideal points.

$$\mathbf{l} = \mathbf{x}_{Id_1} \times \mathbf{x}_{Id_2} = \begin{bmatrix} m_1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} m_2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ m_1 \\ m_2 \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Normal to the plane $w=0$.

Set of the directions of all lines in the plane.

Projective Transformations in 2D

Projective Transformations

- *2D projective geometry:*
Study of properties of the projective plane P^2 that are invariant under a group of transformations called projectivities.
- *Projectivity:*
Invertible mapping $h : P^2 \times P^2$ that maps lines to lines: if $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are collinear, then $h(\mathbf{x}_1), h(\mathbf{x}_2), h(\mathbf{x}_3)$ are also collinear.
- Synonyms for projectivity:
collineation, projective transformation, homography.

Projective Transformations

- Algebraic Formulation -

- Homogeneous matrix H:

$$h(\mathbf{x}) = \mathbf{H}\mathbf{x}$$

$$\mathbf{x}' = \mathbf{H}\mathbf{x}$$

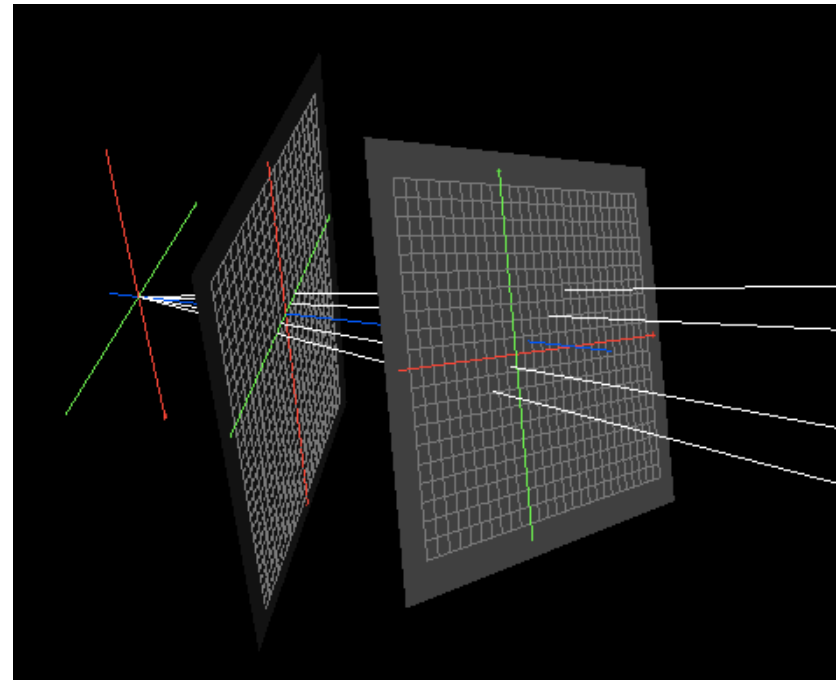
$$\begin{array}{c}
 \boxed{x_1}' \\
 \boxed{x_2}' \\
 \boxed{x_3}'
 \end{array}
 =
 \begin{array}{c}
 \boxed{h_{11}} \\
 \boxed{h_{21}} \\
 \boxed{h_{31}}
 \end{array}
 \begin{array}{c}
 h_{12} \\
 h_{22} \\
 h_{32}
 \end{array}
 \begin{array}{c}
 h_{13} \\
 h_{23} \\
 h_{33}
 \end{array}
 \begin{array}{c}
 \boxed{x_1} \\
 \boxed{x_2} \\
 \boxed{x_3}
 \end{array}$$

- 8 DOF (up to a scale factor)

Projective Transformations

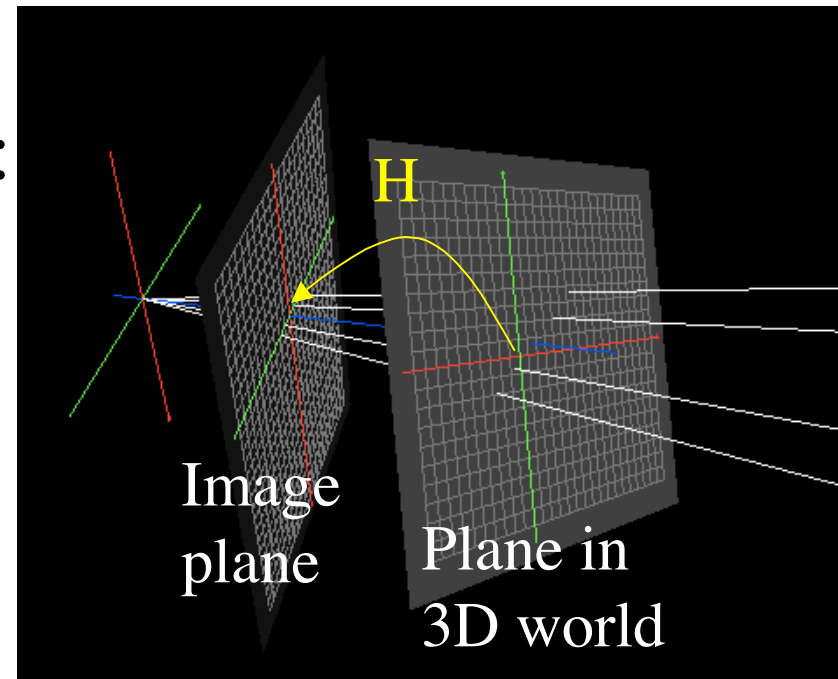
- Central Projection -

- Central projection between 2 planes maps lines to lines.
- Not preserved:
 - Parallel lines
 - Angles



Removing Projective Distortion from a Perspective Image of a Plane

- Distortion due to projective projection H : Parallel lines in 3D converge to a finite point in a projected image.
- Undo distortion by computing H^{-1} .



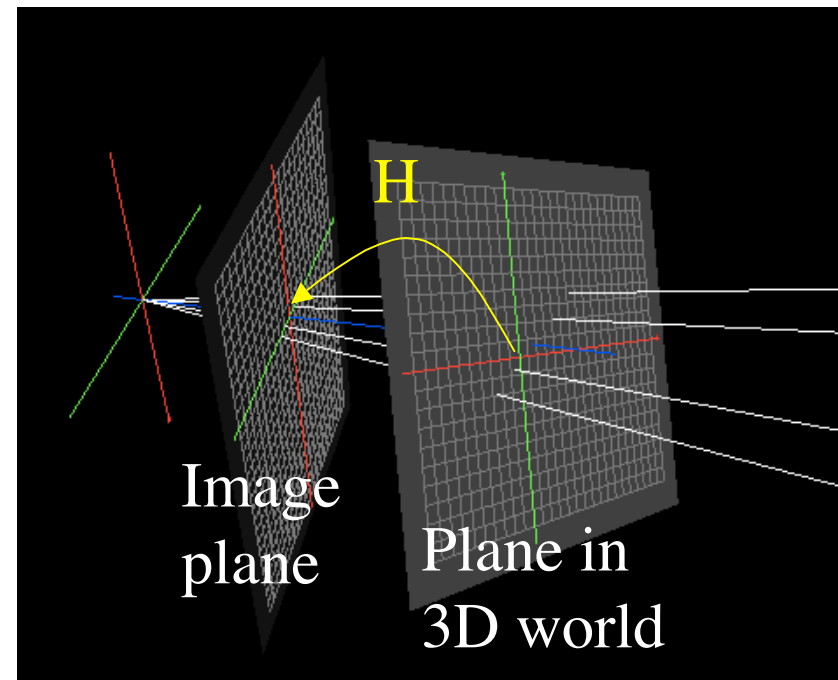
Removing Projective Distortion from a Perspective Image of a Plane

- Match 4 non-collinear world $(x_{wrlld}, y_{wrlld}, 1)_{i=1..4}$ and image $(x_{img}, y_{img})_{i=1..4}$ points.
- Set of 8 linear equations:

$$x_{img} = \frac{h_{11}x_{wrlld} + h_{12}y_{wrlld} + h_{13}}{h_{31}x_{wrlld} + h_{32}y_{wrlld} + h_{33}}$$

$$y_{img} = \frac{h_{21}x_{wrlld} + h_{22}y_{wrlld} + h_{23}}{h_{31}x_{wrlld} + h_{32}y_{wrlld} + h_{33}}$$

- Solve for 8 parameters of H.



Transformation Hierarchy

Hierarchy of Transformations

- Hierarchy:
Projective, affine, Euclidean (isometry, similarity)
- Properties:
Degrees of freedom, invariants

Hierarchy of Transformations

- Class I: Isometries -

- Description: rotation followed by a translation; preserves Euclidean distance.
- Transformation matrix: ($\theta = \pm 1$)

$$\begin{bmatrix} \theta \cos \theta & -\sin \theta & t_x \\ \sin \theta & \theta \cos \theta & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} \\ \mathbf{0}^T \\ 1 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ 1 \end{bmatrix}$$

- DOF: 3
- Invariants: length, angles, area.

Hierarchy of Transformations

- Class II: Similarities -

- Description: isometry plus scaling; preserves shape, *metric structure*.

- Transformation matrix:

$$\begin{bmatrix} s \cos \theta & -s \sin \theta & t_x \\ s \sin \theta & s \cos \theta & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} sR & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

- DOF: 4
- Invariants: angles, parallel lines, ratios of lengths and areas.

Hierarchy of Transformations

- Class III: Affinities -

- Description: similarity plus skew (non-isotropic scaling (λ_1, λ_2)).

- Transformation matrix:

$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} A & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

- DOF: 6
- Invariants: parallel lines, ratio of lengths of parallel line segments, ratio of areas.

Hierarchy of Transformations

- Class IV: Projectivities -

- Description: non-singular linear transformation of homogeneous coordinates.
- Transformation matrix:

$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ v_1 & v_2 & 1 \end{bmatrix} = \begin{bmatrix} A \\ \mathbf{v}^T \\ 1 \end{bmatrix}$$

- DOF: 8
- Invariants: cross ratio of 4 collinear points (cross ratio of lengths of a line).

Hierarchy of Transformations

- Affine vs. Projective -

- For a given affinity,
 - area scaling is the same everywhere
 - orientation of a line is independent of its location in the image
- For a projectivity,
 - area scaling varies with position (distant objects look smaller)
 - orientation depends on location (parallel lines converge at the vanishing point)
 - vector \mathbf{v} in third row of H

Hierarchy of Transformations

- Affine vs. Projective -

Mapping of an ideal point $(x_1, x_2, 0)^T$:

- Affine:

$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \\ 0 \end{bmatrix}$$

- Projective:

$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ v_1 & v_2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \\ v_1x_1 + v_2x_2 \end{bmatrix}$$

- The ideal point doesn't stay at infinity! (It becomes a *vanishing point*).

Hierarchy of Transformations

- Decomposition of a Projectivity -

- A projectivity can be decomposed into a chain of transformations:

$$H = H_S H_A H_P$$

$$= \begin{bmatrix} sR & \mathbf{t} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^T & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0}^T & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{v}^T & v \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^T & v \end{bmatrix}$$

- \mathbf{K} is an upper-triangular matrix, normalized as $\det \mathbf{K} = 1$.
- Valid if $v \neq 0$, unique if $s > 0$.

Recovery of Affine and Metric Properties from Images

Recovery of Affine and Metric Properties from Images

- Projectivity: 8 DOF
- Affinity: 6 DOF
- Similarity: 4 DOF
- Go from projectivity to similarity (recover 4 DOF), by using
 - the line at infinity, \mathbf{l} (2 DOF)
 - two projected right angles (2 DOF)

Recovery of Affine Properties

- The Line at Infinity, \mathbf{l} -

- Under projective transformations, \mathbf{l} is mapped to the vanishing line (connecting several vanishing points).
- Find vanishing points in the image by identifying intersections of projected parallel lines.
- Find H to transform the projected \mathbf{l} (horizon line) back to its canonical position, $\mathbf{l} = (0, 0, 1)^T$.

Recovery of Metric Properties

- Projected right angles -

- Under affine projection, angles are not invariant.
- Find two projected right angles in the image and “unskew” them.
- Find H to transform the lines forming the projected right angles back to their canonical position.

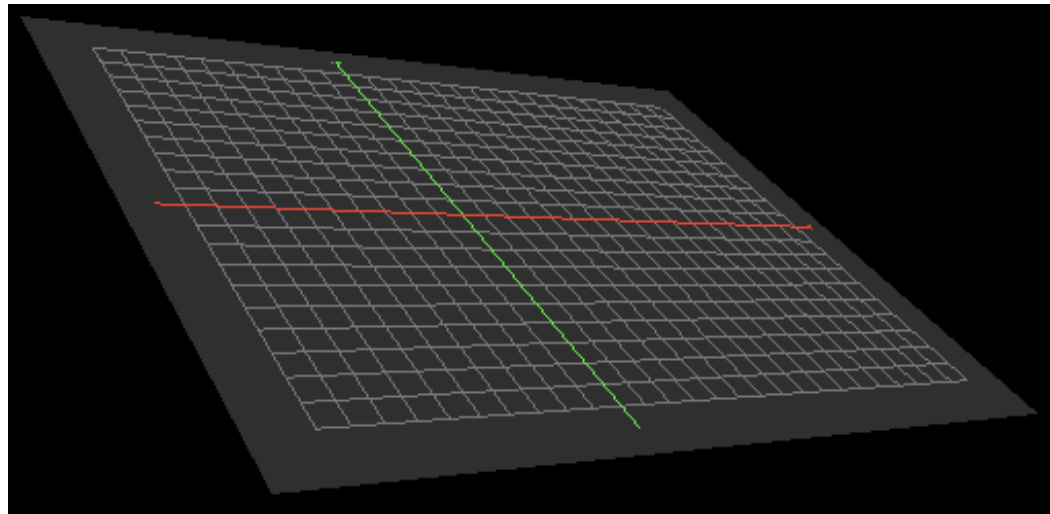
Part 2:
Projective Geometry and
Transformations in 3D

Literature

- Richard Hartley and Andrew Zisserman, “Multiple View Geometry in Computer Vision”, Cambridge University Press, 2000. (Sections 2, 3)
- Illustrations (VRML)
at [illustrations/ProjGeom2D/](#)
and [illustrations/ProjGeom3D/](#)

2. Projective Geometry and Transformations in 3D

- Geometric distortion due to perspective Projection
- *Invariant*:
 - Straight lines
- Not invariant:
 - Angles
 - Parallel lines
- Points, lines, and planes

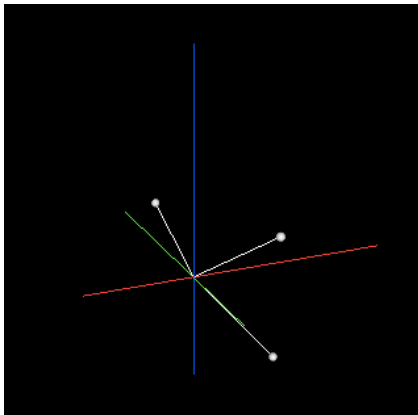


3D Projective Space

- Points -

- *Inhomogeneous* notation in \mathbf{R}^3

$$P = (x, y, z)^T$$



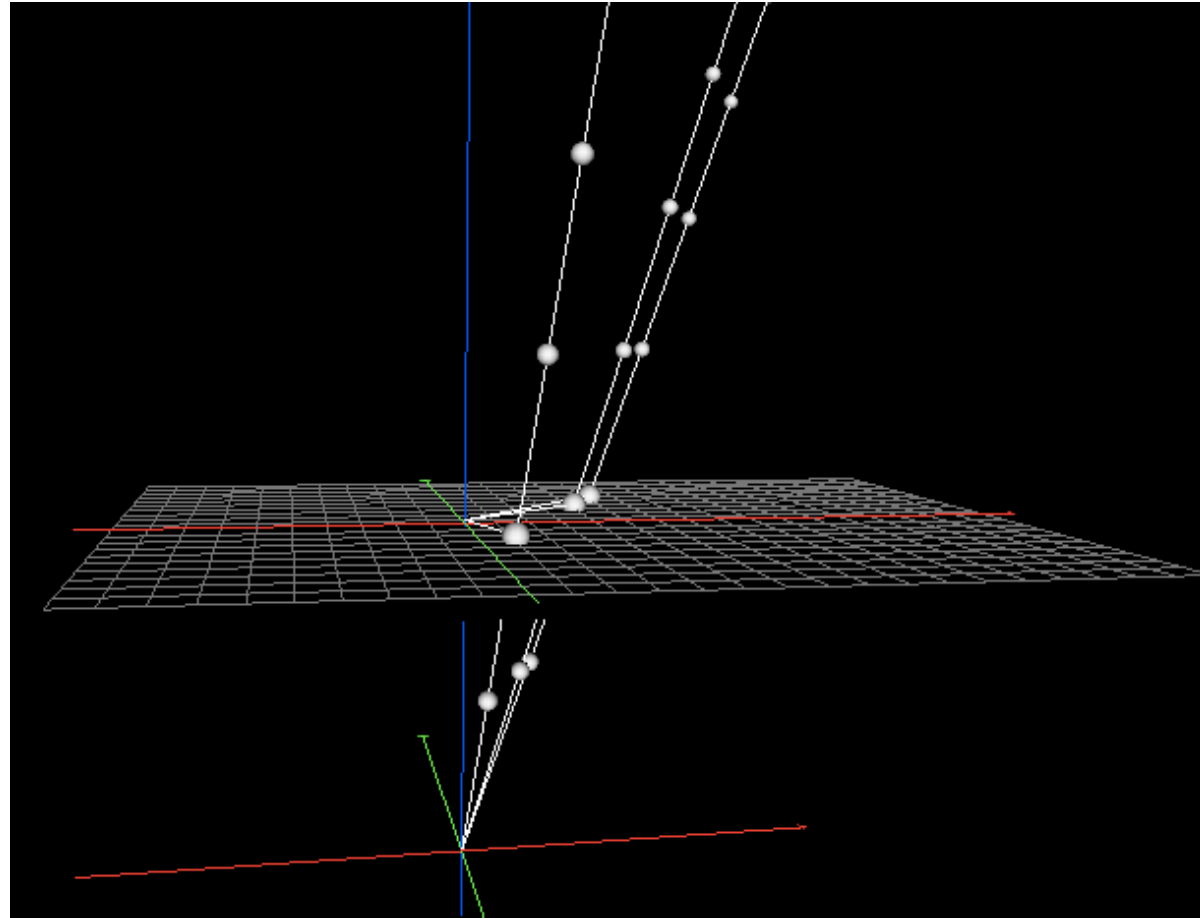
- *Homogeneous* notation in \mathbf{P}^3 (*projective space*)

$$\begin{aligned} \mathbf{x} &= (wx, wy, wz, w)^T \\ &= w(x, y, z, 1)^T \end{aligned}$$

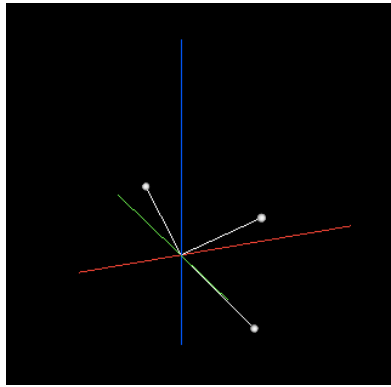
REMINDER
(illustration from P2)

2D Projective Plane

- Points -



$$\mathbf{x} = w(x, y, 1)^T$$

 \mathbb{R}^3

Examples

 \mathbb{P}^3

$$P_1 = (2, -4, -3)^T$$

$$P_2 = (-1, 2, 3)^T$$

$$P_3 = (2.5, -3, -3)^T$$

$$\mathbf{x}_1 = w(2, -4, -3, 1.0)^T$$

$$\mathbf{x}_2 = w(-1, -2, 3, 1.0)^T$$

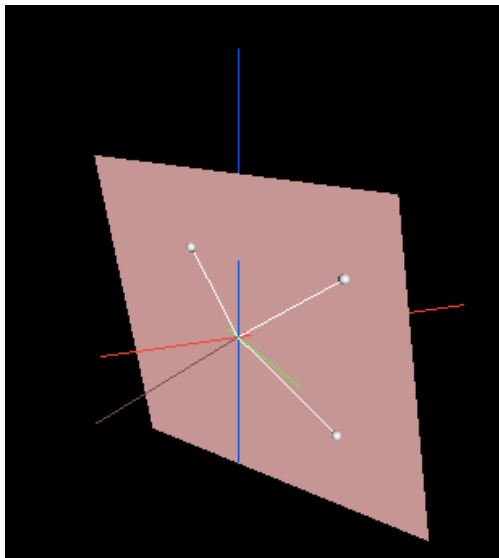
$$\mathbf{x}_3 = w(2.5, -3, -3, 1.0)^T$$

3D Projective Space

- Planes -

- Plane equation: $ax + by + cz + d = 0$
Plane normal: $\mathbf{n} = (a, b, c)/|\mathbf{n}|$

- Homogeneous plane notation in P^3 (projective space): $\square = k(a, b, c, d)^T$



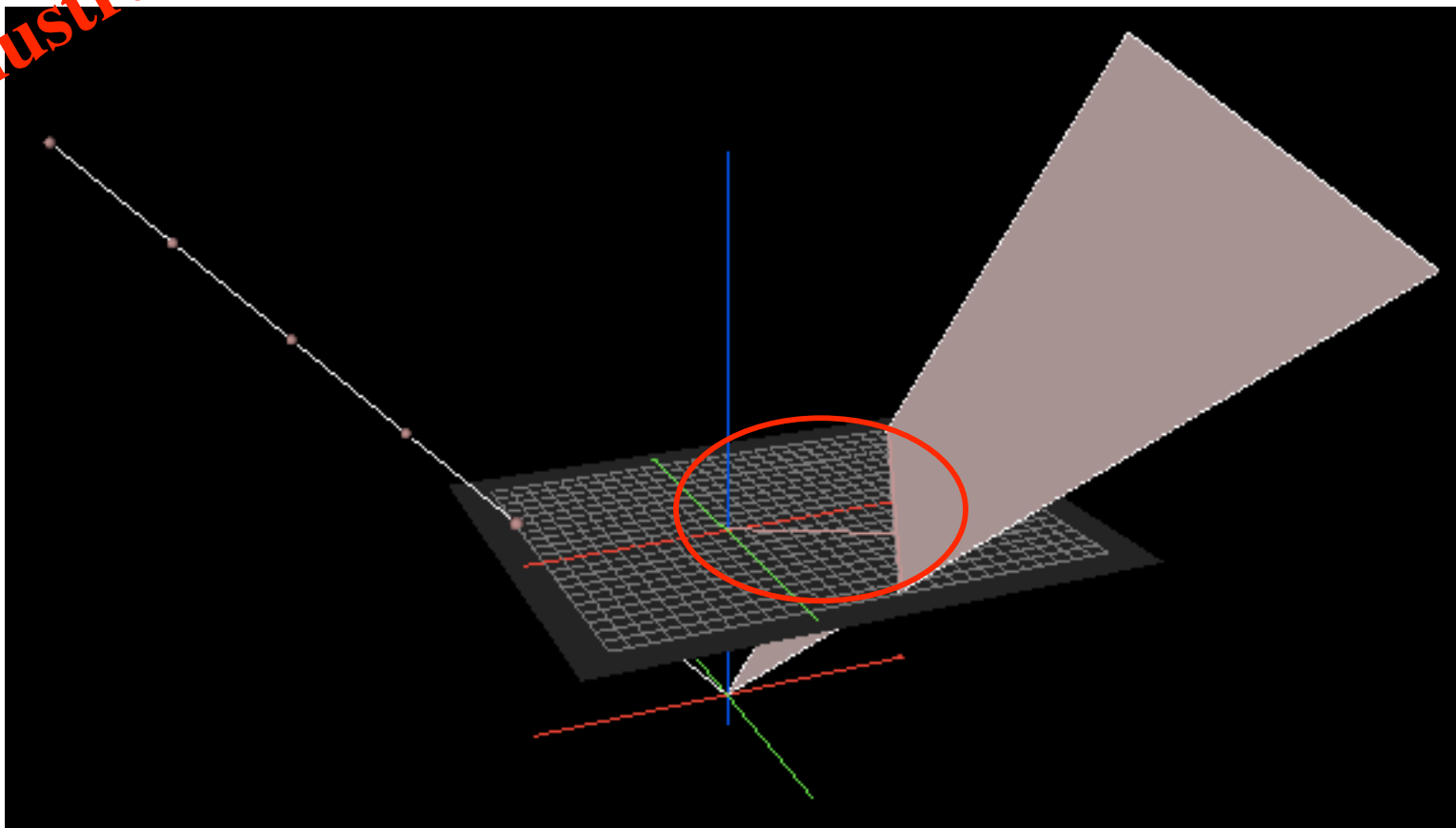
$$w(x, y, z, 1) \cdot k \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0$$

$$\mathbf{x}^T \square = \square^T \mathbf{x} = 0$$

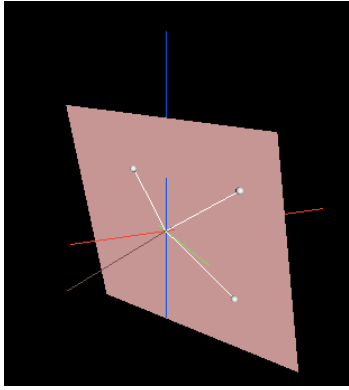
REMINDER
(illustration from P2)

2D Projective Plane

- Lines -



$$\mathbf{l} = k(a, b, c)^T$$

 \mathbb{R}^3

Example

 \mathbb{P}^3

$$\square : 6x + 3y + z - 3 = 0$$

$$\square = k(6, 3, 1, -3)^T$$

3D Projective Space

- Points vs. Planes -

- A plane is defined by the join of 3 points.

$$\begin{matrix} \square & \mathbf{x}_1^T & \square \\ \square & & \square \\ \square & \mathbf{x}_2^T & \square \\ \square & & \square \\ \square & \mathbf{x}_3^T & \square \\ \square & & \square \end{matrix} \begin{matrix} \square \\ \square \\ \square \\ \square \\ \square \\ \square \end{matrix} = 0$$

- A point is defined by the intersection of three planes.
- Duality: $\mathbf{x}^T \square = \square^T \mathbf{x} = 0$

3D Projective Space

- Column Spaces and Nullspaces -

- For 3 points:
(3 planes similarly)

$$\begin{array}{cccc}
 \boxed{x_1} & y_1 & z_1 & 1.0 \boxed{} \\
 \boxed{x_2} & y_2 & z_2 & 1.0 \boxed{} \\
 \boxed{x_3} & y_3 & z_3 & 1.0 \boxed{}
 \end{array} = 0$$

3D-column space
 (rank =3)

1D-nullspace

3 points

normal to plane

3D Projective Space

- Column Spaces and Nullspaces -

- 3 collinear points: (linearly dependent)
(or 3 planes intersecting in a line)

$$\begin{array}{cccc}
 \boxed{x_1} & y_1 & z_1 & 1.0 \boxed{} \\
 \boxed{x_2} & y_2 & z_2 & 1.0 \boxed{} \\
 \boxed{x_3} & y_3 & z_3 & 1.0 \boxed{}
 \end{array} = 0$$

2D-column space 2D-nullspace

(rank =2)

line

line

3D Projective Space

- Lines -

- Join of two points or
Intersection of two planes
- 4 degrees of freedom (DOF)
(homogeneous 5-vector)
- Representations:
 - Join of two points (nullspace and span)
 - Intersection of two planes (Plücker matrices)
 - Map between both (Plücker line coordinates)

Projective Transformations

Projective Transformations

- Algebraic Formulation -

- Homogeneous matrix H:

$$h(\mathbf{x}) = H\mathbf{x}$$

$$\mathbf{x}' = H\mathbf{x}$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

- 15 DOF (up to a scale factor)

Hierarchy of Transformations

- Hierarchy:
Projective, affine, Euclidean (isometry, similarity)
- Properties:
Degrees of freedom, invariants

Hierarchy of Transformations

- Class I: Isometries -

- Description: rotation followed by a translation; preserves Euclidean distance.
- Transformation matrix:

$$\begin{bmatrix} \mathbf{R}_{3 \times 3} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

- DOF: 6
- Invariants: volume.

Hierarchy of Transformations

- Class II: Similarities -

- Description: isometry plus scaling; preserves shape, *metric structure*.
- Transformation matrix:

$$\begin{bmatrix} \mathbf{R}_{3 \times 3} \\ \mathbf{0}^T \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ 1 \end{bmatrix}$$

- DOF: 7
- Invariants: angles, parallel lines, ratios of lengths and areas, absolute conic.

Hierarchy of Transformations

- Class III: Affinities -

- Description: similarity plus skew (non-isotropic scaling (λ_1, λ_2)).
- Transformation matrix:

$$\begin{bmatrix} \lambda_1 A & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

- DOF: 12
- Invariants: Parallelism of planes, volume ratios, centroids, plane at infinity.

Hierarchy of Transformations

- Class IV: Projectivities -

- Description: non-singular linear transformation of homogeneous coordinates.
- Transformation matrix:

$$\begin{bmatrix} \square & \mathbf{A} & \mathbf{t} \\ \square & & \\ \square & \mathbf{v}^T & v \end{bmatrix}$$

- DOF: 15
- Invariants: Intersection and tangency of surfaces in contact, sign of Gaussian curvature.

3D Projective Space - Plane at Infinity -

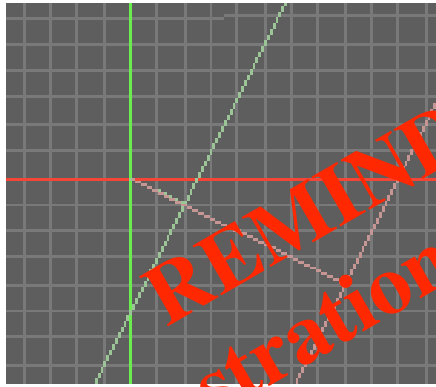
REMINDER
(illustration from P2)

2D Projective Plane - Ideal Points -

- Intersection of parallel lines $\mathbf{l}_1 = (a,b,c)^T$ and $\mathbf{l}_2 = (a,b,c')^T$

$$\mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2 = (c' - c) \begin{bmatrix} b \\ a \\ 0 \end{bmatrix}$$

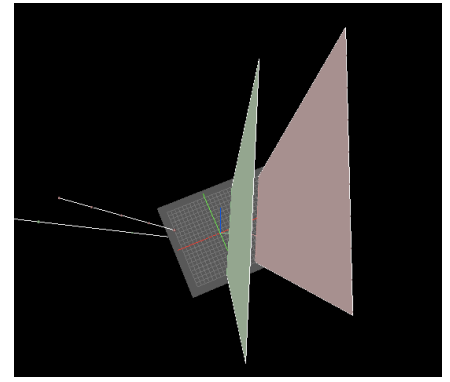
- Parallel lines intersect “at infinity”.
- *Ideal points* lie on plane $w=0$
(Points at infinity).



REMINDER
(Illustration from P2)

\mathbb{R}^2

Example



\mathbb{P}^2

$$l_1 : 2x - y - 2 = 0$$

$$l_2 : 2x - y - 0.5 = 0$$

$$l_2 = (2, -1, -2)^T$$

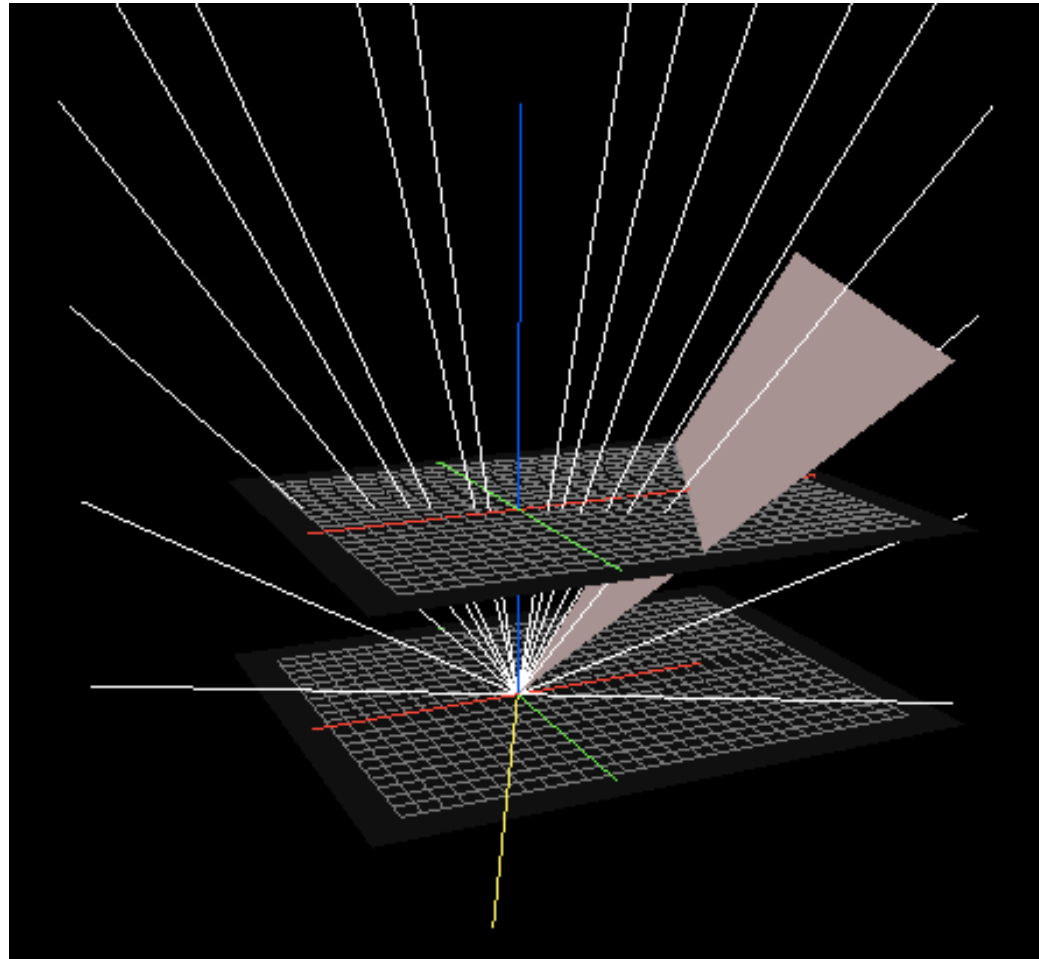
$$l_3 = (2, -1, -0.5)^T$$

$$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1.5 \end{bmatrix} = 1.5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

REMINDER
(illustration from P2)

2D Projective Plane

- Ideal Points -



2D Projective Plane

- Line at Infinity -

REMINDER
(illustration from P2)

Set of all ideal points (*points at infinity*):

$$\mathbf{x}_{Id_i} = (x_i, y_i, 0)^T = s(x_i/y_i, 1, 0)^T$$

i.e.: all ideal points lie in plane, $w = 0$.

- The *line at infinity* represents all ideal points.

$$\mathbf{l} = \mathbf{x}_{Id_1} \times \mathbf{x}_{Id_2} = \begin{bmatrix} m_1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} m_2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ m_1 \\ m_2 \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Normal to the plane $w=0$.

Set of the directions of all lines in the plane.

3D Projective Space

- Plane at Infinity -

- Two parallel planes intersect along a line at infinity.
- A line is parallel to another line, iff their point of intersection is on a line at infinity.
- Differently oriented planes form (together with parallel planes) different lines at infinity.
- All lines at infinity are coplanar. They lie within the *plane at infinity* at $w = 0$.
- It is represented by the vector

$$\square = (0, 0, 0, 1)^T$$

Hierarchy of Transformations

REMINDER
(illustration from P2)

Affine vs. Projective -

Mapping of an ideal point $(x_1, x_2, 0)^T$:

- Affine:

$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \\ 0 \end{bmatrix}$$

- Projective:

$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ v_1 & v_2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \\ v_1x_1 + v_2x_2 \end{bmatrix}$$

- The ideal point doesn't stay at infinity! (It becomes a *vanishing point*).

Estimation

- 2D Projective Transformations -

Estimation

- 2D Projective Transformations -

- Minimum number of measurements:
4 points
- Minimal solutions for robust estimation algorithms (RANSAC)
- Approximate solutions for noisy data:
minimize a cost function
 - algebraic error
 - geometrical or statistical image distance

Estimation

- Direct Linear Transformation (DLT)-

- Given: at least four 2D-to-2D point correspondences
- Transformation: $\mathbf{x}_i' = \mathbf{H}\mathbf{x}_i$

- Notation:

$$H = \begin{bmatrix} \mathbf{h}^{1T} \\ \mathbf{h}^{2T} \\ \mathbf{h}^{3T} \end{bmatrix}$$

$$= \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix}$$

Estimation

- Direct Linear Transformation (DLT)-

- Given: at least four 2D-to-2D point correspondences
- Transformation: $\mathbf{x}_i' = H\mathbf{x}_i$
- Observation: $\mathbf{x}_i' \propto H\mathbf{x}_i = 0$

$$\mathbf{x}_i' \begin{bmatrix} \mathbf{h}^{1T} \mathbf{x}_i \\ \mathbf{h}^{2T} \mathbf{x}_i \\ \mathbf{h}^{3T} \mathbf{x}_i \end{bmatrix} = 0$$

$$\begin{bmatrix} y_i' \mathbf{h}^{3T} \mathbf{x}_i - w_i' \mathbf{h}^{2T} \mathbf{x}_i \\ w_i' \mathbf{h}^{1T} \mathbf{x}_i - x_i' \mathbf{h}^{3T} \mathbf{x}_i \\ x_i' \mathbf{h}^{2T} \mathbf{x}_i - y_i' \mathbf{h}^{1T} \mathbf{x}_i \end{bmatrix} = 0$$

$$\mathbf{x}_i' = \begin{bmatrix} x_i' \\ y_i' \\ z_i' \end{bmatrix}$$

Estimation

- Direct Linear Transformation (DLT)-

• Resort:

$$\begin{bmatrix} y_i' \mathbf{h}^{3T} \mathbf{x}_i & w_i' \mathbf{h}^{2T} \mathbf{x}_i \\ w_i' \mathbf{h}^{1T} \mathbf{x}_i & x_i' \mathbf{h}^{3T} \mathbf{x}_i \\ x_i' \mathbf{h}^{2T} \mathbf{x}_i & y_i' \mathbf{h}^{1T} \mathbf{x}_i \end{bmatrix} = 0$$

$$\begin{bmatrix} \mathbf{0}^T & w_i' \mathbf{x}_i^T & y_i' \mathbf{x}_i^T \\ w_i' \mathbf{x}_i^T & \mathbf{0}^T & x_i' \mathbf{x}_i^T \\ y_i' \mathbf{x}_i^T & x_i' \mathbf{x}_i^T & \mathbf{0}^T \end{bmatrix} \begin{bmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{bmatrix} = 0$$

$$A_i \mathbf{h} = 0$$

Estimation

- Direct Linear Transformation (DLT)-

$$\begin{bmatrix}
 0 & 0 & 0 & w_i' x_i & w_i' y_i & w_i' z_i & y_i' x_i & y_i' y_i & y_i' z_i \\
 w_i' x_i & w_i' y_i & w_i' z_i & 0 & 0 & 0 & x_i' x_i & x_i' y_i & x_i' z_i \\
 y_i' x_i & y_i' y_i & y_i' z_i & x_i' x_i & x_i' y_i & x_i' z_i & 0 & 0 & 0
 \end{bmatrix}
 \begin{bmatrix}
 h_1 \\
 h_2 \\
 h_3 \\
 h_4 \\
 h_5 \\
 h_6 \\
 h_7 \\
 h_8 \\
 h_9
 \end{bmatrix}
 = \mathbf{0}$$

Estimation

- Direct Linear Transformation (DLT)-

$$\begin{bmatrix} 0 \\ w_i' \mathbf{x}_i \end{bmatrix} \mathbf{0}^T \quad \begin{bmatrix} w_i' \mathbf{x}_i \\ 0 \end{bmatrix} \mathbf{x}_i^T \quad \begin{bmatrix} y_i' \mathbf{x}_i \\ x_i' \mathbf{x}_i \end{bmatrix} \begin{bmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{bmatrix} = \mathbf{0}$$
$$A_i \mathbf{h} = \mathbf{0}$$

Estimation

Direct Linear Estimation (DLT)

- 2 equations per point
- 8 unknowns (plus scale factor)
- --> at least four points to solve for H
computing the SVD of $A = UDV^T$
 - overdetermined case:
minimize a suitable cost function, while $\mathbf{h} \neq \mathbf{0}$
 - inhomogeneous solution ($h_9 = 1$)
 - degenerate configurations (collinear points)

Estimation

Cost Functions

- Algebraic distance: $\|A\mathbf{h}\|^2$
 $d_{alg}(\mathbf{x}_1, \mathbf{x}_2)^2 = a_1^2 + a_2^2$
 $\mathbf{a} = (a_1, a_2, a_3) = \mathbf{x}_1 \square \mathbf{x}_2$
- Geometric distance: *transfer error*
Euclidean distance in 2. image between
measured and projected point
- Reprojection error, both images:
symmetric transfer error

Robust Estimation

- *Outliers versus inliers*
 - model fitting (minimization of some cost function)
 - data classification
- Select a “good” set of samples
 - margin of tolerance

Robust Estimation

- RANSAC -

RANdOm SAmpLe Consensus

- Select a *random sample* of points to compute an initial estimation of H.
- Compute *support* for this estimation i.e., number of inliers (measurements within tolerance): *consensus set*
- Repeat with several random samples.

Robust Estimation

- RANSAC -

- What distance threshold?
 - Assume Gaussian noise
 - Use Chi-Square test (95%)
- How many samples?

Ensure that with high probability (99%) at least one of the random samples is free of outliers.
- How large is a consensus set?

Similar to the number of inliers “believed” to be in the data set.

