

Introduction to Conics

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Conics

Curve described by 2nd-degree equation in the plane

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

or *homogenized* $x \mapsto x_1/x_3, y \mapsto x_2/x_3$

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

or in matrix form

$$\mathbf{x}^T \mathbf{C} \mathbf{x} = 0 \quad \text{with} \quad \mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

5DOF: ratio of 6 parameters $\{a:b:c:d:e:f\}$

Five Points Define a Conic

For each point the conic passes through

$$ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f = 0$$

or

$$\begin{bmatrix} x_i^2 & x_iy_i & y_i^2 & x_i & y_i & 1 \end{bmatrix} \mathbf{c} = 0 \quad \mathbf{c} = [a, b, c, d, e, f]^T$$

stacking constraints yields

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \mathbf{c} = 0$$

Conics

Conics (conic sections) can be defined as intersection of a double cone (canonical equation: $z=x^2+y^2$) with a plane passing through the apex:



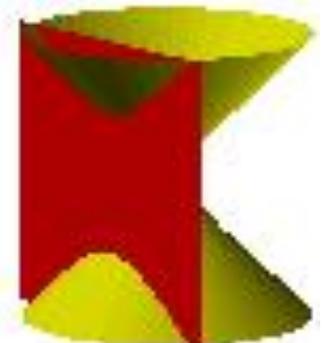
Circle



Ellipse



Parabola



Hyperbola

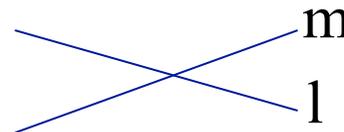
$$\mathbf{x}^T \mathbf{C} \mathbf{x} = 0 \quad \text{with} \quad \mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

Degenerate Conics

A conic is degenerate if matrix \mathbf{C} is not of full rank

e.g. two lines (rank 2)

$$\mathbf{C} = \mathbf{l}\mathbf{m}^T + \mathbf{m}\mathbf{l}^T$$



e.g. repeated line (rank 1)

$$\mathbf{C} = \mathbf{l}\mathbf{l}^T$$



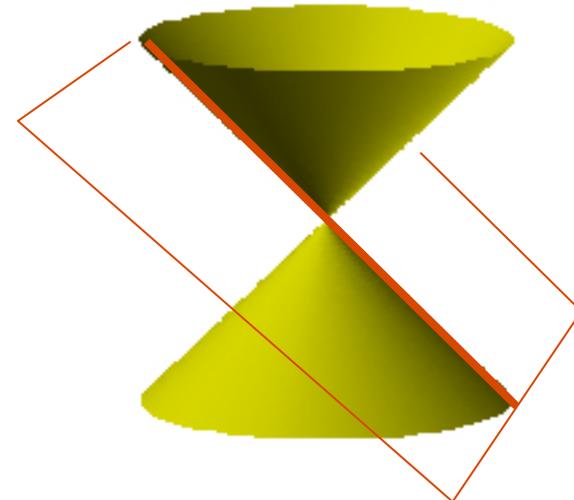
Degenerate Conics



Degenerate conic section: intersecting lines

intersecting lines (rank 2)

$$C = lm^T + ml^T$$

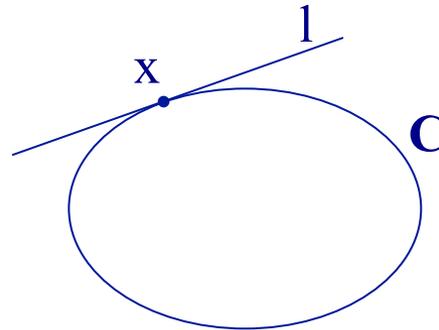


repeated line (rank 1)

$$C = ll^T$$

Tangent Lines to Conics

The line l tangent to C at point x on C is given by $l=Cx$



$$\left. \begin{array}{l} l^t x = x^t C x = \phi \\ l^t y = y^t C y = \phi \end{array} \right\} \Rightarrow \begin{array}{l} (\alpha x + \beta y)^t C (\alpha x + \beta y) = \phi \\ l \subset C \text{ (degenerate conic)} \end{array}$$

Dual Conics

A line tangent to the conic C satisfies $1^T C^* 1 = 0$

$$x^T C C^* C x = \phi$$

In case C is full rank (up to scale): $C^* = C^{-1}$

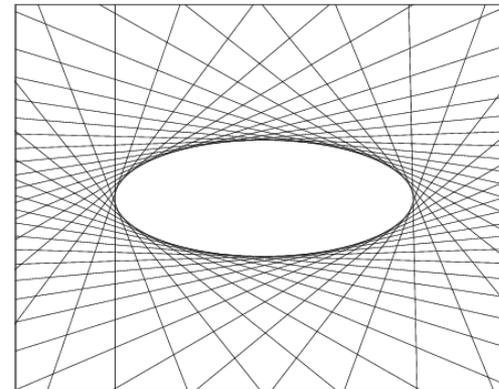
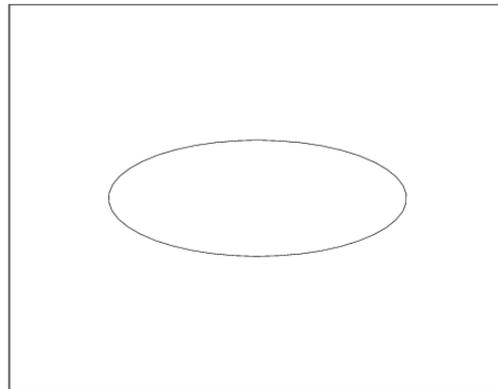
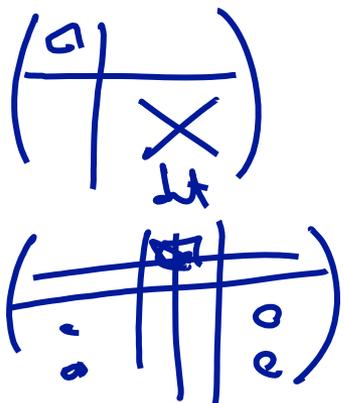
$$x^T C x = \phi$$

If not C^* is the matrix of cofactors:

$$C_{ij}^* = (-1)^{i+j} \det(\hat{C}_{ij})$$

Definition :

$$A^{-1} = \frac{1}{\det(A)} (\text{Adjoint of } A) = \frac{1}{\det(A)} (\text{cofactor matrix of } A)^t$$



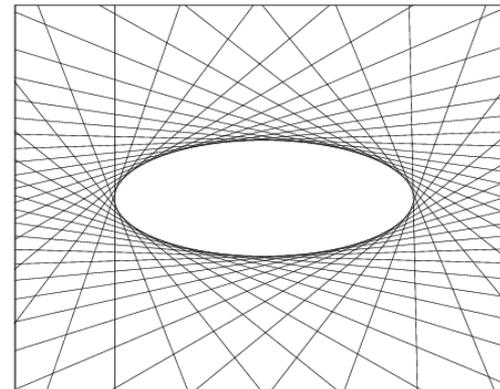
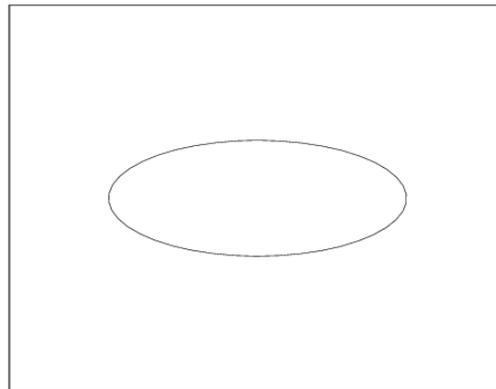
Dual Conics

A line tangent to the conic \mathbf{C} satisfies $\mathbf{1}^T \mathbf{C}^* \mathbf{1} = 0$

In case \mathbf{C} is full rank: $\mathbf{C}^* = \mathbf{C}^{-1}$

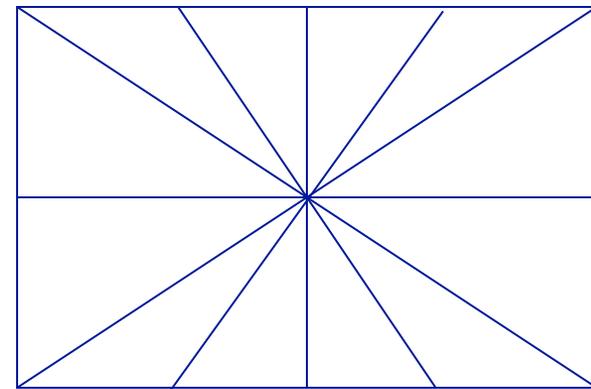
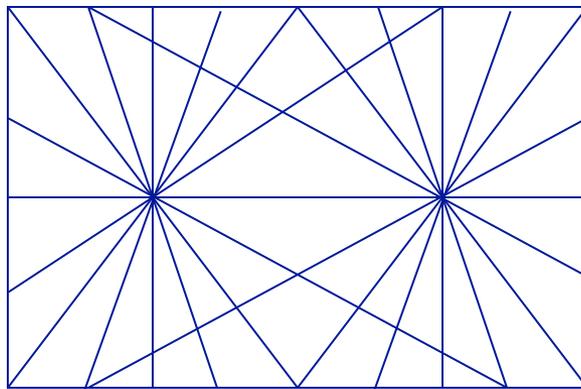
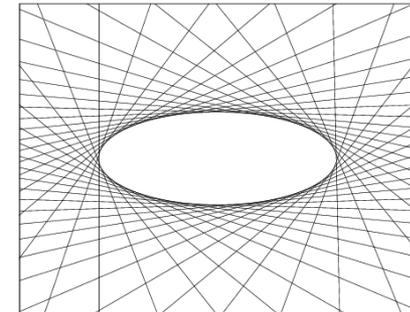
If not \mathbf{C}^* is the matrix of cofactors: $\mathbf{C}_{ij}^* = (-1)^{i+j} \det(\hat{\mathbf{C}}_{ij})$

Dual conics = line conics = conic envelopes



Degenerate Dual Conics

A line tangent to the conic C satisfies $l^T C^* l = 0$



$$C^* = X_1 X_2^T + X_2 X_1^T$$

Handwritten: $l^T C^* l = l^T X_1 X_2^T l + l^T X_2 X_1^T l$

$$C^* = X_1 X_1^T$$

Degenerate line conics: 2 points (rank 2), double point (rank 1)

Note that for degenerate conics $(C^*)^* \neq C$

Transformation of Conics

Transformation of a point by the homography H: $\mathbf{X}' = \mathbf{H}\mathbf{X}$

Transformation of a conic C by the homography H: $\mathbf{C}' = \mathbf{H}^{-\top}\mathbf{C}\mathbf{H}^{-1}$

$$\mathbf{X}^{\top}\mathbf{C}\mathbf{X} = 0 \rightarrow \mathbf{X}'^{\top}\mathbf{C}'\mathbf{X}' = 0$$

$$\mathbf{X}^{\top}\mathbf{H}^{\top}\mathbf{C}'\mathbf{H}\mathbf{X} = 0$$

$$\mathbf{C} = \mathbf{H}^{\top}\mathbf{C}'\mathbf{H}$$

$$\mathbf{C}' = \mathbf{H}^{-\top}\mathbf{C}\mathbf{H}^{-1}$$

*Additional Material Not Covered in the Lecture

Transformation of Conics

Transformation of a line by the homography H : $\mathbf{l}' = \mathbf{H}^{-T} \mathbf{l}$

Transformation of a dual conic C^* by the homography H : $C'^* = \mathbf{H} \mathbf{C}^* \mathbf{H}^T$

$$\mathbf{l}^T \mathbf{C}^* \mathbf{l} = 0 \rightarrow \mathbf{l}'^T \mathbf{C}'^* \mathbf{l}' = 0$$

$$\mathbf{l}^T \mathbf{H}^{-1} \mathbf{C}'^* \mathbf{H}^{-T} \mathbf{l} = 0$$

$$\mathbf{C}^* = \mathbf{H}^{-1} \mathbf{C}'^* \mathbf{H}^{-T}$$

$$\mathbf{C}'^* = \mathbf{H} \mathbf{C}^* \mathbf{H}^T$$

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The Circular Points

$$\mathbf{I} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \quad \mathbf{J} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

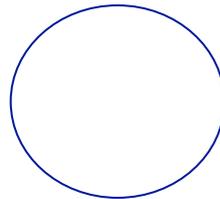
$$\mathbf{I}' = \mathbf{H}_S \mathbf{I} = \begin{bmatrix} s \cos \theta & -s \sin \theta & t_x \\ s \sin \theta & s \cos \theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = s e^{-i\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \mathbf{I}$$

The circular points \mathbf{I} , \mathbf{J} are fixed points under the projective transformation \mathbf{H} iff \mathbf{H} is a similarity

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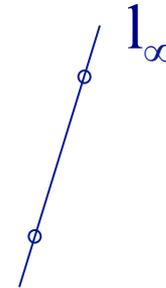
The Circular Points

“circular points”



$$x_1^2 + x_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

$$x_3 = 0$$



$$x_1^2 + x_2^2 = 0$$

$$I = (1, i, 0)^T$$

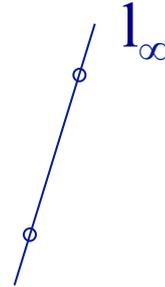
$$J = (1, -i, 0)^T$$

In Euclidean a circle is defined with 3 points.

In projective conics are defined with 5 points, for circles these are 3 points plus the two circular points.

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The Circular Points



$$x_1^2 + x_2^2 = 0$$

$$I = (1, i, 0)^T$$

$$J = (1, -i, 0)^T$$

Algebraically, encode orthogonal directions

$$I = (1, 0, 0)^T + i(0, 1, 0)^T$$

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Conic Dual to the Circular Points

$$\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} (1 \ -i \ 0) + \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} (1 \ i \ 0)$$

$$\mathbf{C}_\infty^* = \mathbf{I}\mathbf{J}^\top + \mathbf{J}\mathbf{I}^\top$$

$$\mathbf{C}_\infty^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{C}_\infty^* = \mathbf{H}_S \mathbf{C}_\infty^* \mathbf{H}_S^\top$$

The dual conic \mathbf{C}_∞^* is fixed conic under the projective transformation \mathbf{H} iff \mathbf{H} is a similarity

Note: \mathbf{C}_∞^* has 4DOF
 \mathbf{l}_∞ is the nullvector

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Conic Dual to the Circular Points

$$\mathbf{C}_{\infty}^* = \mathbf{I}\mathbf{J}^T + \mathbf{J}\mathbf{I}^T$$

$$\mathbf{C}_{\infty}^* = \mathbf{H}_S \mathbf{C}_{\infty}^* \mathbf{H}_S^T$$

$$\mathbf{C}_{\infty}^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note: \mathbf{C}_{∞}^* has 4DOF

\mathbf{l}_{∞} is the nullvector

$$\mathbf{I}^T \mathbf{l}_{\infty} = \mathbf{J}^T \mathbf{l}_{\infty} = 0$$

$$\mathbf{C}_{\infty}^* \mathbf{l}_{\infty} = \mathbf{I}\mathbf{J}^T \mathbf{l}_{\infty} + \mathbf{J}\mathbf{I}^T \mathbf{l}_{\infty} = 0$$

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Conic Dual to the Circular Points

$$\mathbf{C}_{\infty}^* = \mathbf{I}\mathbf{J}^T + \mathbf{J}\mathbf{I}^T$$

$$\mathbf{C}_{\infty}^* = \mathbf{H}_S \mathbf{C}_{\infty}^* \mathbf{H}_S^T$$

$$\mathbf{C}_{\infty}^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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Note: \mathbf{C}_{∞}^* has 4DOF

\mathbf{l}_{∞} is the nullvector

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Measurement of Angles

Euclidean Space: $\mathbf{l} = (l_1, l_2, l_3)^\top$ $\mathbf{m} = (m_1, m_2, m_3)^\top$

$$\cos \theta = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}$$

We can use the conic dual to the circular points (\mathbf{C}_∞^*) in order to measure angles in projective space

Projective Space: $\cos \theta = \frac{\mathbf{l}^\top \mathbf{C}_\infty^* \mathbf{m}}{\sqrt{(\mathbf{l}^\top \mathbf{C}_\infty^* \mathbf{l})(\mathbf{m}^\top \mathbf{C}_\infty^* \mathbf{m})}}$

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Measurement of Angles

Projective Space:
$$\cos \theta = \frac{\mathbf{l}^T \mathbf{C}_\infty^* \mathbf{m}}{\sqrt{(\mathbf{l}^T \mathbf{C}_\infty^* \mathbf{l})(\mathbf{m}^T \mathbf{C}_\infty^* \mathbf{m})}}$$

This is invariant to projective transformation \mathbf{H} :

$$\mathbf{X} \rightarrow \mathbf{HX}$$

$$\mathbf{l}^T \mathbf{C}_\infty^* \mathbf{m} \rightarrow \mathbf{l}^T \mathbf{H}^{-1} \mathbf{H} \mathbf{C}_\infty^* \mathbf{H}^T \mathbf{H}^{-T} \mathbf{m} = \mathbf{l}^T \mathbf{C}_\infty^* \mathbf{m}$$

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Measurement of Angles

Euclidean Space: $\mathbf{l} = (l_1, l_2, l_3)^\top$ $\mathbf{m} = (m_1, m_2, m_3)^\top$

$$\cos \theta = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}$$

We can use the conic dual to the circular points (\mathbf{C}_∞^*) in order to measure angles in projective space

Projective Space: $\cos \theta = \frac{\mathbf{l}^\top \mathbf{C}_\infty^* \mathbf{m}}{\sqrt{(\mathbf{l}^\top \mathbf{C}_\infty^* \mathbf{l})(\mathbf{m}^\top \mathbf{C}_\infty^* \mathbf{m})}}$

$$\mathbf{l}^\top \mathbf{C}_\infty^* \mathbf{m} = 0 \quad (\text{orthogonal})$$

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Decomposition of Projective Transformations

$$\mathbf{H} = \mathbf{H}_S \mathbf{H}_A \mathbf{H}_P = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{v}^\top & v \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^\top & v \end{bmatrix}$$

$$\mathbf{A} = s\mathbf{R}\mathbf{K} + \mathbf{t}\mathbf{v}^\top$$

decomposition unique (if chosen $s > 0$)

\mathbf{K} upper-triangular, $\det \mathbf{K} = 1$

Example:

$$\mathbf{H} = \begin{bmatrix} 1.707 & 0.586 & 1.0 \\ 2.707 & 8.242 & 2.0 \\ 1.0 & 2.0 & 1.0 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 2 \cos 45^\circ & -2 \sin 45^\circ & 1.0 \\ 2 \sin 45^\circ & 2 \cos 45^\circ & 2.0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

***Additional Material Not Covered in the Lecture in 2012**

Metric Properties from Images

$$\begin{aligned}
 \mathbf{C}_{\infty}^{*'} &= (\mathbf{H}_P \mathbf{H}_A \mathbf{H}_S) \mathbf{C}_{\infty}^* (\mathbf{H}_P \mathbf{H}_A \mathbf{H}_S)^{\top} \\
 &= (\mathbf{H}_P \mathbf{H}_A) \mathbf{H}_S \mathbf{C}_{\infty}^* \mathbf{H}_S^{\top} (\mathbf{H}_P \mathbf{H}_A)^{\top} \\
 &= (\mathbf{H}_P \mathbf{H}_A) \mathbf{C}_{\infty}^* (\mathbf{H}_P \mathbf{H}_A)^{\top} \\
 &= \begin{bmatrix} \mathbf{K} \mathbf{K}^{\top} & \mathbf{K}^{\top} \mathbf{v} \\ \mathbf{v}^{\top} \mathbf{K} & \mathbf{v}^{\top} \mathbf{v} \end{bmatrix}
 \end{aligned}$$

Rectifying transformation by applying the SVD

$$\mathbf{C}_{\infty}^{*'} = \mathbf{U} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{U}^{\top} \quad \mathbf{H} = \mathbf{U}$$

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Projective Conic Classification

- for any conic C we can find a decomposition such that it contains only the entries 1,-1,0 in this order

$$\mathbf{C} = \mathbf{U}\mathbf{D}\mathbf{U}^T \quad \mathbf{D} = \mathbf{U}^{-1}\mathbf{C}\mathbf{U}^{-T}$$
$$\mathbf{D} = \text{diag}(s_1, s_2, s_3) \text{diag}(e_1, e_2, e_3) \text{diag}(s_1, s_2, s_3) \quad e_i = \pm 1 \text{ or } 0$$

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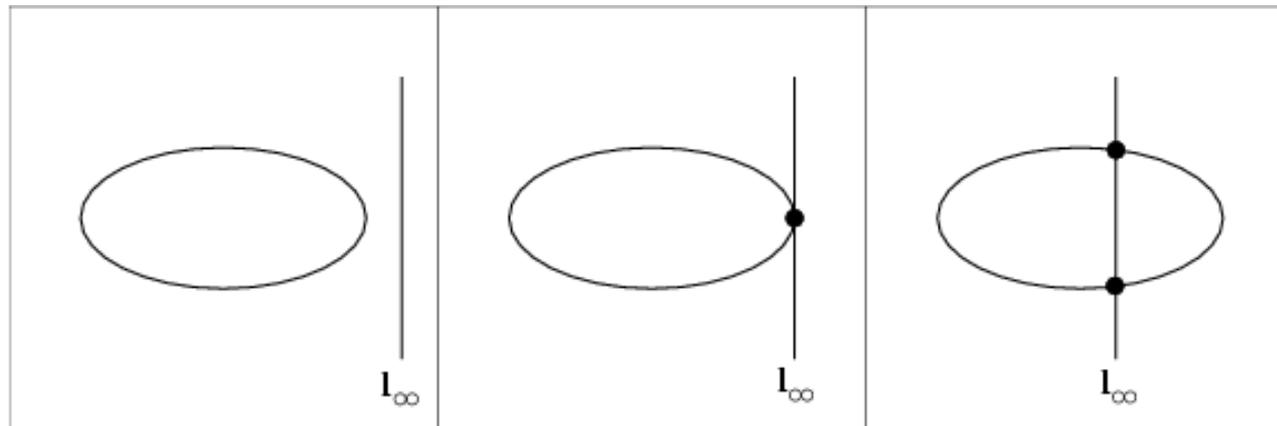
Projective Conic Classification

- This leads us to the following classification

Diagonal	Equation	Conic type
(1,1,1)	$x^2 + y^2 + w^2 = 0$	improper conic-no real point
(1,1,-1)	$x^2 + y^2 - w^2 = 0$	circle
(1,1,0)	$x^2 + y^2 = 0$	single real point (0,0,1)
(1,-1,0)	$x^2 - y^2 = 0$	two lines $x = \pm y$
(1,0,0)	$x^2 = 0$	single line $x=0$ counted twice

*Additional Material Not Covered in the Lecture in 2012

Affine Conic Classification



ellipse

parabola

hyperbola

*Additional Material Not Covered in the Lecture in 2012

Quadrics

*Additional Material Not Covered in the Lecture in 2012

Quadrics and Dual Quadrics

Quadrics: $X^T Q X = 0$ (Q : 4x4 symmetric matrix)

1. 9 d.o.f.

2. in general 9 points define quadric

3. $\det Q=0 \leftrightarrow$ degenerate quadric

4. pole – polar $\pi = QX$

5. (plane \cap quadric)=conic $C = M^T Q M$ $\pi : X = Mx$

6. transformation $Q' = H^{-T} Q H^{-1}$

$$Q = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \\ \circ & \circ & \circ & \bullet \end{bmatrix}$$

Dual Quadrics: $\pi^T Q^* \pi = 0$

1. relation to quadric $Q^* = Q^{-1}$ (non-degenerate)

2. transformation $Q'^* = H Q^* H^T$

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Quadric Classification

Rank	Sign.	Diagonal	Equation	Realization
4	4	(1,1,1,1)	$X^2+ Y^2+ Z^2+1=0$	No real points
	2	(1,1,1,-1)	$X^2+ Y^2+ Z^2=1$	Sphere
	0	(1,1,-1,-1)	$X^2+ Y^2= Z^2+1$	Hyperboloid (1S)
3	3	(1,1,1,0)	$X^2+ Y^2+ Z^2=0$	Single point
	1	(1,1,-1,0)	$X^2+ Y^2= Z^2$	Cone
2	2	(1,1,0,0)	$X^2+ Y^2= 0$	Single line
	0	(1,-1,0,0)	$X^2= Y^2$	Two planes
1	1	(1,0,0,0)	$X^2=0$	Single plane

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