

3D Computer Vision Camera Models II

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based on a course given at UNC by Marc Pollefeys & the book "Multiple View Geometry" by Hartley & Zisserman

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Outline of the Computer Vision Lecture

Projective Geometry and Transformations

- Transformations with homogeneous coordinates
- 2D projective geometry
- 3D projective geometry
- 1D projective geometry

Estimation

- Direct and Iterative Estimation
- DLT Algorithm
- Cost Functions
- Robust Estimation (RANSAC)

Camera Geometry

- Camera Models
- Camera Calibration

Algorithm Evaluation and Error Analysis

- Statistics, Error Propagation

Advanced Estimation

- Bundle Adjustment
- Iterative Closest Point (ICP)

Single-View Geometry

Two-View Geometry

- Epipolar Geometry / Fundamental Matrix / Essential Matrix
- 3D Reconstruction of Cameras and Structure

Three-View Geometry

- Trifocal Tensor
- Computation of the Trifocal Tensor

N-View Geometry

- N-Linearities and Multiple View Tensors
- N-View Computational Methods
- Factorization
- Structure From Motion



Outline – Camera Model

- History of Camera (Camera Obscura)
- Intensity Images
- Geometric Parameters of a Finite camera
- Projective Camera
 - Camera Center, Principal Plane, Axis Plane, Principal Point, Principal Ray
 - Decomposition of Camera Matrix
- Cameras at Infinity
- Other Camera Models (Pushbroom and Line Cameras)

Cameras at Infinity

Cameras with its center lying at infinity

$$P \begin{bmatrix} d \\ 0 \end{bmatrix} = 0 \Rightarrow \det M = 0 \quad M \text{ is singular}$$

Two types of cameras at infinity:
Affine and non-affine cameras

Affine Cameras

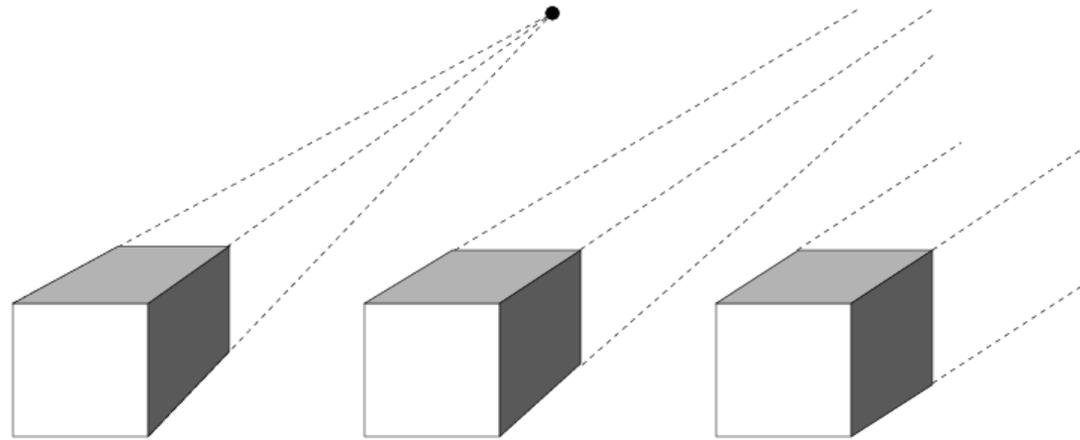
Definition:

An affine camera is a camera with a camera matrix P in which the last row p^{3T} is of the form $(0,0,0,1)^T$.

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

points at infinity are mapped to points at infinity

Affine Cameras



perspective

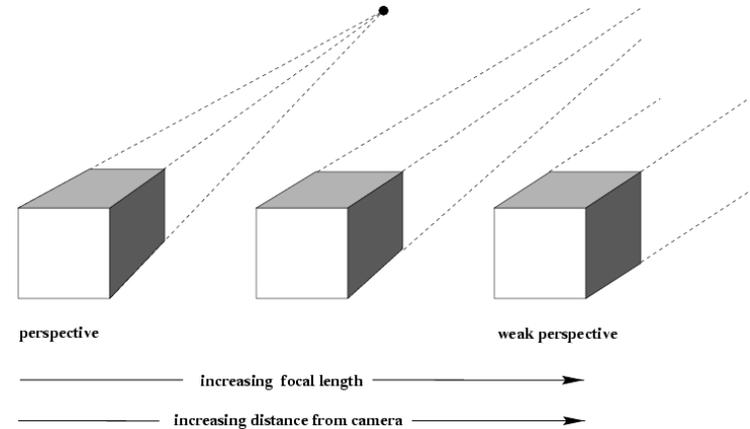
weak perspective

————— increasing focal length —————>
————— increasing distance from camera —————>



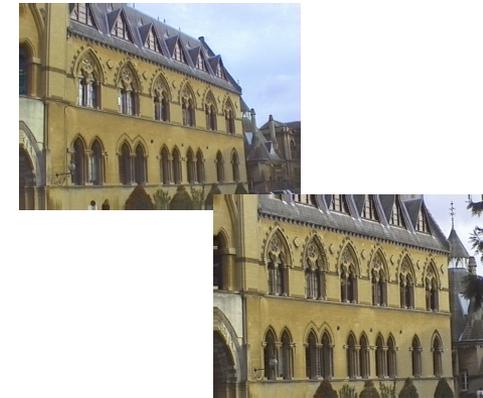
Affine Cameras

$$P_0 = KR \begin{bmatrix} I & -\tilde{C} \end{bmatrix} = K \begin{bmatrix} \mathbf{r}^{1T} & -\mathbf{r}^{1T} \tilde{C} \\ \mathbf{r}^{2T} & -\mathbf{r}^{2T} \tilde{C} \\ \mathbf{r}^{3T} & -\mathbf{r}^{3T} \tilde{C} \end{bmatrix}$$



$d_0 = -\mathbf{r}^{3T} \tilde{C}$ distance of the world origin from the camera center in direction of the principle ray

$$P_t = K \begin{bmatrix} \mathbf{r}^{1T} & -\mathbf{r}^{1T} (\tilde{C} - t\mathbf{r}^3) \\ \mathbf{r}^{2T} & -\mathbf{r}^{2T} (\tilde{C} - t\mathbf{r}^3) \\ \mathbf{r}^{3T} & -\mathbf{r}^{3T} (\tilde{C} - t\mathbf{r}^3) \end{bmatrix} = K \begin{bmatrix} \mathbf{r}^{1T} & -\mathbf{r}^{1T} \tilde{C} \\ \mathbf{r}^{2T} & -\mathbf{r}^{2T} \tilde{C} \\ \mathbf{r}^{3T} & d_t \end{bmatrix}$$



modifying p_{34} corresponds to moving along principal ray

Affine Cameras

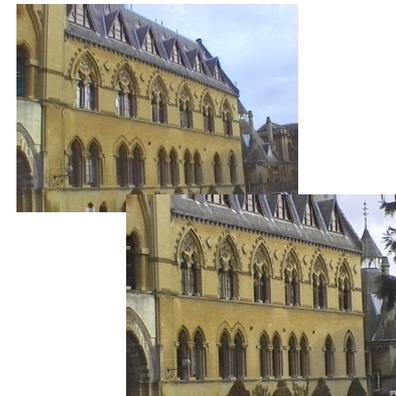
combine tracking back and zooming

$$P_t = K \begin{bmatrix} d_t / d_0 & & & \\ & d_t / d_0 & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} r^{1T} & -r^{1T} \tilde{C} \\ r^{2T} & -r^{2T} \tilde{C} \\ r^{3T} & d_t \end{bmatrix}$$

magnification factor $k=d_t/d_0$
remains fixed

$$= \frac{d_t}{d_0} K \begin{bmatrix} r^{1T} & -r^{1T} \tilde{C} \\ r^{2T} & -r^{2T} \tilde{C} \\ r^{3T} d_0 / d_t & d_0 \end{bmatrix}$$

$$P_\infty = \lim_{t \rightarrow \infty} P_t = K \begin{bmatrix} r^{1T} & -r^{1T} \tilde{C} \\ r^{2T} & -r^{2T} \tilde{C} \\ 0 & d_0 \end{bmatrix}$$



Error in Employing Affine Cameras

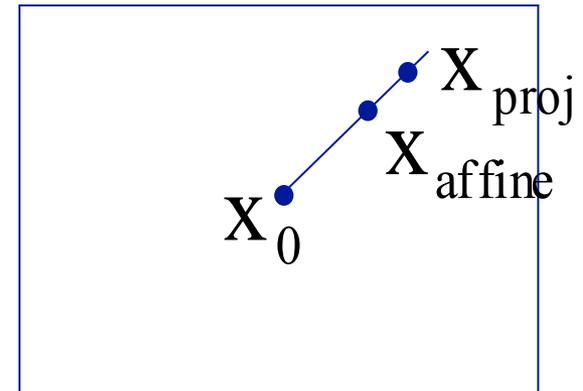
$$\mathbf{X} = \begin{pmatrix} \alpha r^1 + \beta r^2 \\ 1 \end{pmatrix}$$
 point on plane parallel to principal plane and through origin, then

$$\mathbf{P}_0 \mathbf{X} = \mathbf{P}_t \mathbf{X} = \mathbf{P}_\infty \mathbf{X}$$

$$\mathbf{X} = \begin{pmatrix} \alpha r^1 + \beta r^2 + \Delta r^3 \\ 1 \end{pmatrix}$$
 general points (not on the parallel plane) with distance Δ from the plane

$$\mathbf{x}_{\text{proj}} = \mathbf{P}_0 \mathbf{X} = \mathbf{K} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ d_0 + \Delta \end{pmatrix}$$

$$\mathbf{x}_{\text{affine}} = \mathbf{P}_\infty \mathbf{X} = \mathbf{K} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ d_0 \end{pmatrix}$$



Error in Employing Affine Cameras

$$K = \begin{bmatrix} K_{2 \times 2} & \tilde{x}_0 \\ \tilde{0}^T & 1 \end{bmatrix}$$

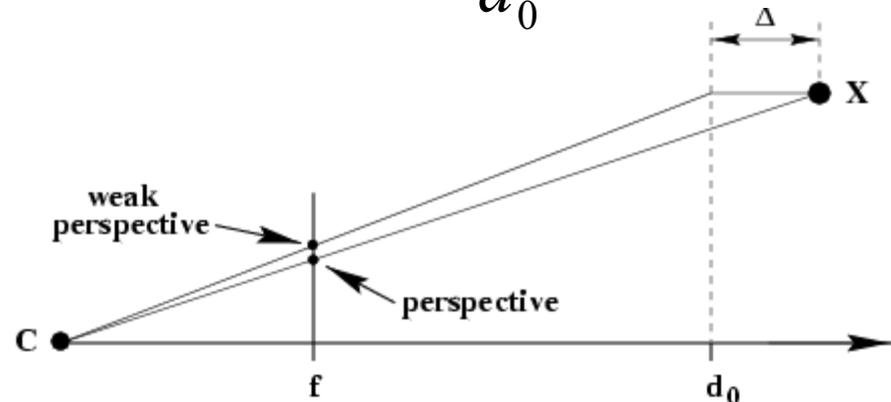
$$X_{\text{proj}} = \begin{pmatrix} K_{2 \times 2} \tilde{X} + (d_0 + \Delta) \tilde{X}_0 \\ d_0 + \Delta \end{pmatrix}$$

$$X_{\text{affine}} = \begin{pmatrix} K_{2 \times 2} \tilde{X} + d_0 \tilde{X}_0 \\ d_0 \end{pmatrix}$$

$$\tilde{X}_{\text{proj}} = \tilde{X}_0 + \frac{K_{2 \times 2} \tilde{X}}{d_0 + \Delta}$$

$$\tilde{X}_{\text{affine}} = \tilde{X}_0 + \frac{K_{2 \times 2} \tilde{X}}{d_0}$$

$$\tilde{X}_{\text{affine}} - \tilde{X}_0 = \frac{d_0 + \Delta}{d_0} (\tilde{X}_{\text{proj}} - \tilde{X}_0)$$



Affine Imaging Conditions

$$\mathbf{X}_{\text{affine}} - \mathbf{X}_{\text{proj}} = \frac{\Delta}{d_0} (\mathbf{X}_{\text{proj}} - \mathbf{X}_0)$$

approximation should only cause small error

1. Δ much smaller than d_0
2. points close to principal ray (i.e. small field of view)

Decomposition of P_∞

$$P_\infty = \begin{bmatrix} K_{2 \times 2} & \tilde{x}_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & \tilde{t} \\ 0 & d_0 \end{bmatrix} = \begin{bmatrix} d_0^{-1} K_{2 \times 2} & \tilde{x}_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & \tilde{t} \\ 0 & 1 \end{bmatrix}$$

absorb d_0 in $K_{2 \times 2}$

$$= \begin{bmatrix} K_{2 \times 2} \tilde{R} & K_{2 \times 2} \tilde{t} + \tilde{x}_0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} K_{2 \times 2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & \tilde{t} + K_{2 \times 2}^{-1} \tilde{x}_0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} K_{2 \times 2} & K_{2 \times 2} \tilde{t} + \tilde{x}_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & 0 \\ 0 & 1 \end{bmatrix}$$

$$P_\infty = \begin{bmatrix} K_{2 \times 2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & \tilde{t} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} K_{2 \times 2} & \tilde{x}_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & 0 \\ 0 & 1 \end{bmatrix}$$

alternatives, because 8dof (3+3+2), not more

Summary of Parallel Projections

$$\mathbf{P}_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

canonical representation

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{2 \times 2} & 0 \\ 0 & 1 \end{bmatrix}$$

calibration matrix

principal point is not defined

Hierarchy of Affine Cameras

$$\mathbf{P}_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{dropping the z-coordinate}$$

orthographic projection

$$\mathbf{H} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix}$$
$$\mathbf{P}_\infty = \begin{bmatrix} \mathbf{r}^{1T} & t_1 \\ \mathbf{r}^{1T} & t_2 \\ 0 & 1 \end{bmatrix} \quad (5\text{dof})$$

Hierarchy of Affine Cameras

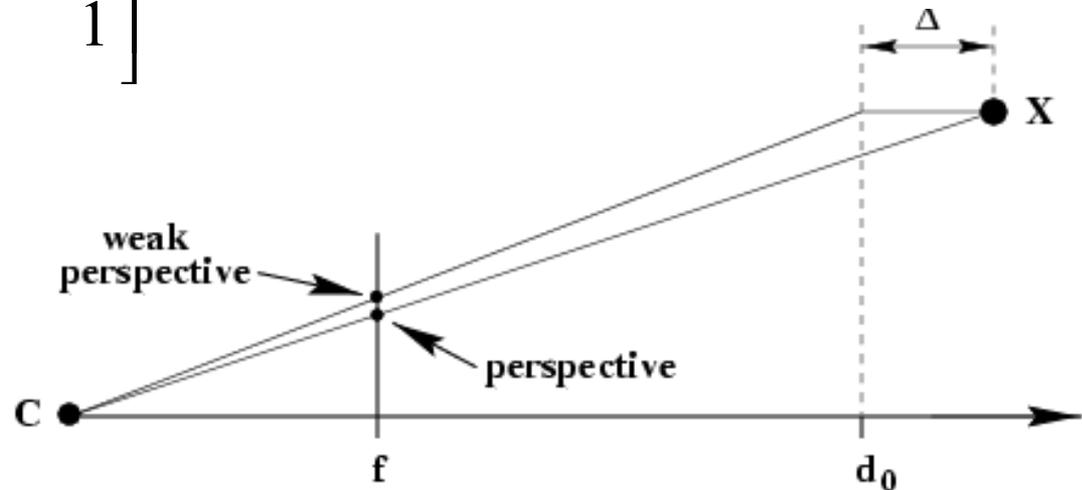
scaled orthographic projection

$$\mathbf{P}_\infty = \begin{bmatrix} k & & & \\ & k & & \\ & & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r}^{1T} & t_1 \\ \mathbf{r}^{1T} & t_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{r}^{1T} & t_1 \\ \mathbf{r}^{1T} & t_2 \\ 0 & 1/k \end{bmatrix} \quad (6\text{dof})$$

Hierarchy of Affine Cameras

weak perspective projection

$$P_{\infty} = \begin{bmatrix} \alpha_x & & & \\ & \alpha_y & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r}^{1T} & t_1 \\ \mathbf{r}^{1T} & t_2 \\ 0 & 1 \end{bmatrix} \quad (7\text{dof})$$



Hierarchy of Affine Cameras

affine camera

$$P_A = \begin{bmatrix} \alpha_x & s & & \\ & \alpha_y & & \\ & & 1 & \\ & & & \end{bmatrix} \begin{bmatrix} \mathbf{r}^{1T} & t_1 \\ \mathbf{r}^{1T} & t_2 \\ 0 & 1 \end{bmatrix} \quad (8\text{dof})$$

$$P_A = \begin{bmatrix} m_{11} & m_{12} & m_{13} & t_1 \\ m_{21} & m_{22} & m_{23} & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

full generality of an affine camera

$$P_A = \left[\begin{array}{c} 3 \times 3 \text{ affine} \end{array} \right] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{c} 4 \times 4 \text{ affine} \end{array} \right]$$

- Affine camera is a projective camera with principal plane at infinity
- Affine camera maps parallel world lines to parallel image lines
- No center of projection, but direction of projection $P_A D=0$

General Camera at Infinity

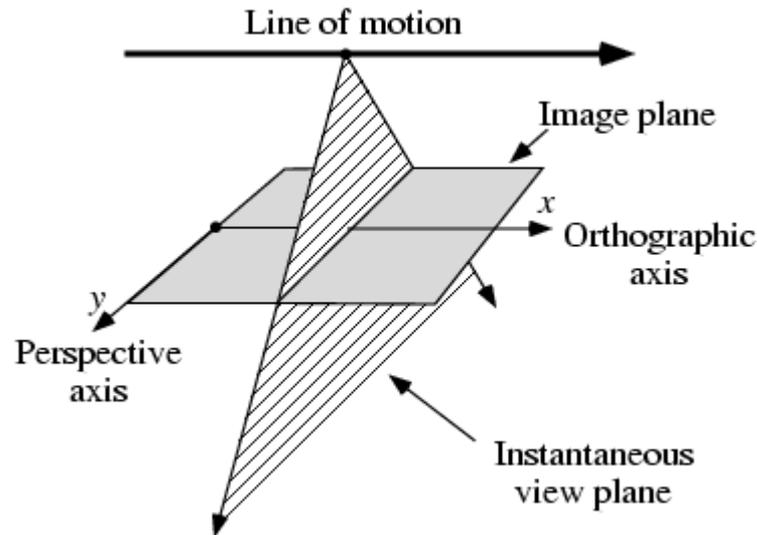
$$P = [M | p_4] \quad M \text{ is singular, but last row not zero}$$

- camera center is on plane at infinity
- principal plane is not plane at infinity
- images of points at infinity are in general not mapped to infinity on the image plane

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Pushbroom Cameras



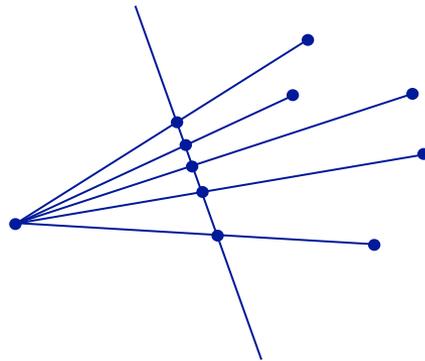
(11dof)

$$X = (X, Y, X, T)^T \quad PX = (x, y, w)^T \quad (x, y/w)^T$$

Straight lines are not mapped to straight lines!
(otherwise it would be a projective camera)

Line Cameras

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad (5\text{dof})$$



Null-space $PC=0$ yields camera center

Also decomposition $\mathbf{P}_{2 \times 3} = \mathbf{K}_{2 \times 2} \mathbf{R}_{2 \times 2} [\mathbf{I}_{2 \times 2} \mid -\tilde{\mathbf{c}}]$

Summary Camera Models

- photometric and radiometric properties of a camera
- geometric parameters of a finite camera
- projective cameras
 - camera anatomy (camera center, principle plane, principle point, and principle axis)
 - camera matrix decomposition (camera center, orientation, and intrinsic parameter)
- cameras at infinity
 - affine cameras
 - non – affine cameras
- alternative models (pushbroom cameras, line cameras)

Literature on Camera Models

- Chapter 6 in R. Hartley and A. Zisserman, “*Multiple View Geometry*”, 2nd edition, Cambridge University Press, 2003.
- Chapter 3 in O. Faugeras, “*Three-dimensional Computer Vision*”, MIT Press, 1993.
- Chapter 2 in E. Trucco and A. Verri, “*Introductory Techniques for 3-D Computer Vision*”, Prentice Hall, 1998.
- H. Gernsheim, “*The Origins of Photography*”, *Thames and Hudson*, 1982.
- A. Shashua. “*Geometry and Photometry in 3D Visual Recognition*”, Ph.D. Thesis, MIT, Nov. 1992. AITR-1401.

Action of Projective Camera on Points

Forward Projection

maps a point in space on the image plane $\mathbb{P}^3 \rightarrow \mathbb{P}^2$

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

vanishing points $\mathbf{D} = (\mathbf{d}^T, 0)^T$

$$\mathbf{x} = \mathbf{P}\mathbf{D} = \left[\mathbf{M} \mid \mathbf{p}_4 \right] \mathbf{D} = \mathbf{M}\mathbf{d}$$

only \mathbf{M} effects the projection of vanishing points

Action of Projective Camera on Points

Back-projection to Rays

$$\mathbf{X} = \mathbf{P}^+ \mathbf{x} \quad \mathbf{P}^+ = \mathbf{P}^\top (\mathbf{P}\mathbf{P}^\top)^{-1} \quad \mathbf{P}\mathbf{P}^+ = \mathbf{I}$$

(pseudo-inverse)

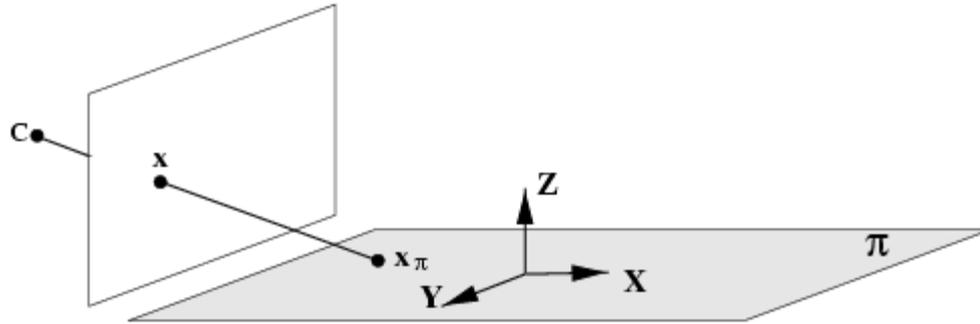
$$\mathbf{P}\mathbf{C} = \mathbf{0} \quad \text{camera center } \mathbf{C}$$

ray is the line formed by those two points

points on the reconstructed ray

$$\mathbf{X}(\lambda) = \mathbf{P}^+ \mathbf{x} + \lambda \mathbf{C}$$

Action of Projective Camera on Planes



$$\mathbf{x} = \mathbf{P}\mathbf{X} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_4 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

The most general transformation that can occur between a scene plane and an image plane under perspective imaging is a plane projective transformation

(affine camera-affine transformation)

Action of Projective Camera on Lines

forward projection

$$X(\mu) = P(A + \mu B) = PA + \mu PB = a + \mu b$$

back-projection

$$\Pi = P^T \mathbf{1}$$

$$\Pi^T X = \mathbf{1}^T P X = 0$$

Action of Projective Camera on Conics

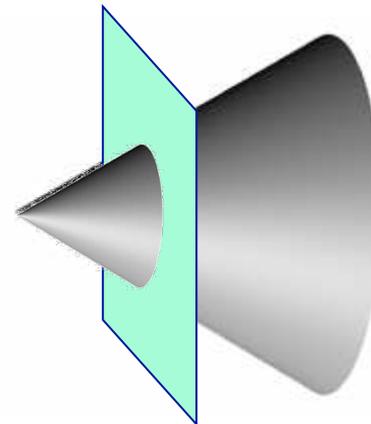
back-projection to cone

$$\mathbf{x}^T \mathbf{C} \mathbf{x} = \mathbf{X}^T \mathbf{P}^T \mathbf{C} \mathbf{P} \mathbf{X} = 0$$

$$\mathbf{Q}_{\text{co}} = \mathbf{P}^T \mathbf{C} \mathbf{P}$$

Note: vertex of degenerate quadric is optical center, since

$$\mathbf{Q}_{\text{co}} \mathbf{C} = \mathbf{P}^T \mathbf{C} (\mathbf{P} \mathbf{C}) = 0$$

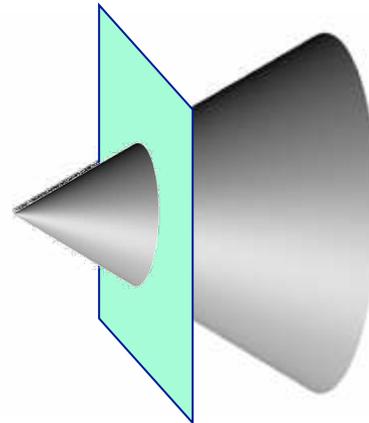


*Additional Material Not Covered in the Lecture

Action of Projective Camera on Conics

back-projection to cone

$$Q_{co} = P^T C P$$



Example $P = K[I \mid 0]$:

$$Q_{co} = \begin{bmatrix} K^T \\ 0 \end{bmatrix} C^T [K \mid 0] = \begin{bmatrix} K^T C K & 0 \\ 0 & 0 \end{bmatrix}$$

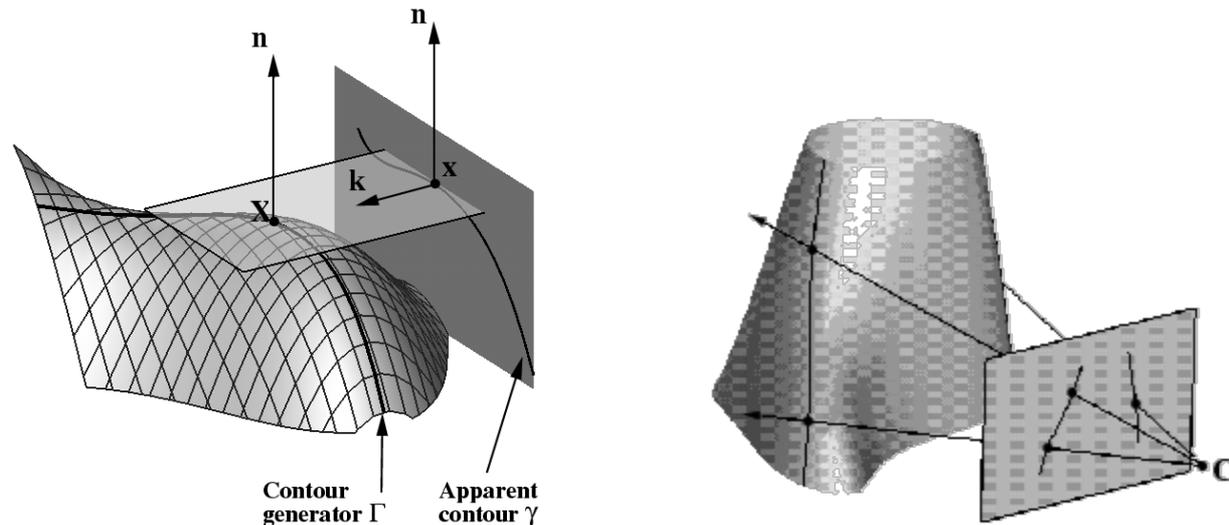
where $C = (0,0,0,1)^T$

*Additional Material Not Covered in the Lecture

Images of Smooth Surfaces

The contour generator Γ is the set of points X on S at which rays are tangent to the surface. The corresponding apparent contour γ is the set of points x which are the image of X , i.e. γ is the image of Γ

The contour generator Γ depends only on position of projection center, γ depends also on rest of P

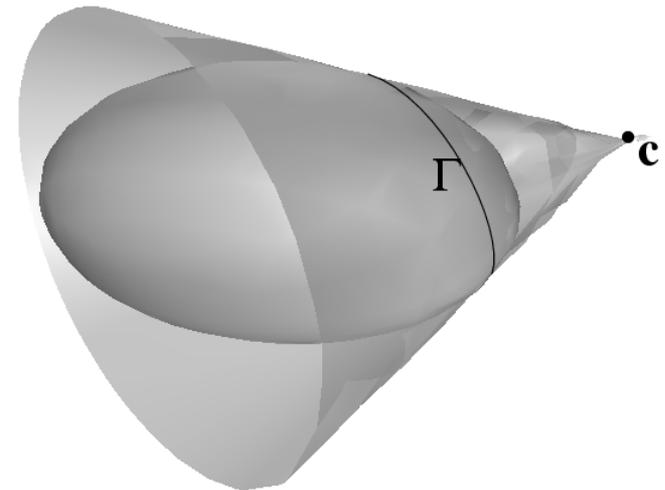


*Additional Material Not Covered in the Lecture

Action of projective camera on quadrics

- Forward projection of quadrics:

$$C^* = PQ^*P^T$$

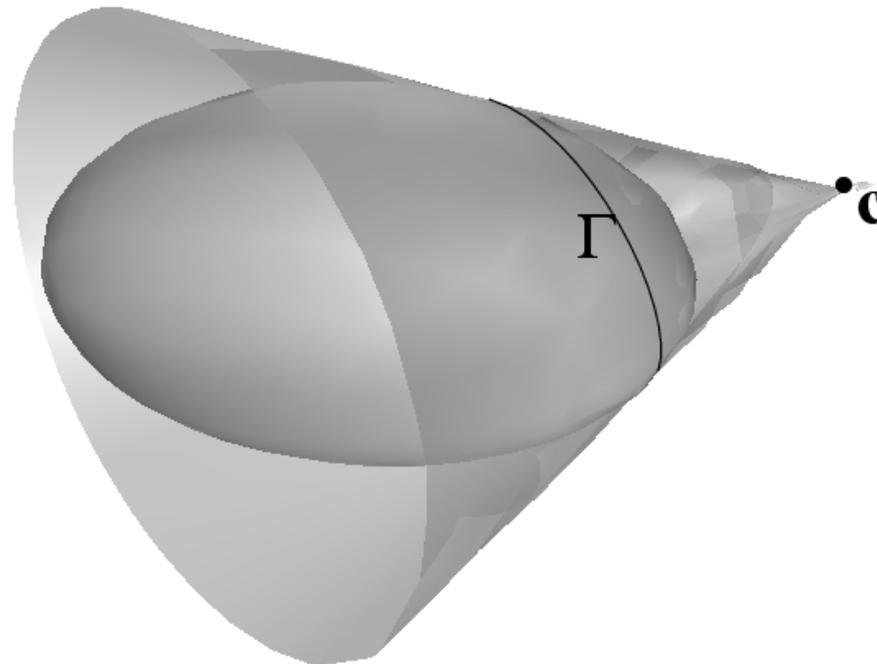


- The plane of G for a quadric Q and camera center C is given by $P=QC$ (follows from pole-polar relation)

*Additional Material Not Covered in the Lecture

Pole-Polar Relationship for Planes and points with respect to quadrics

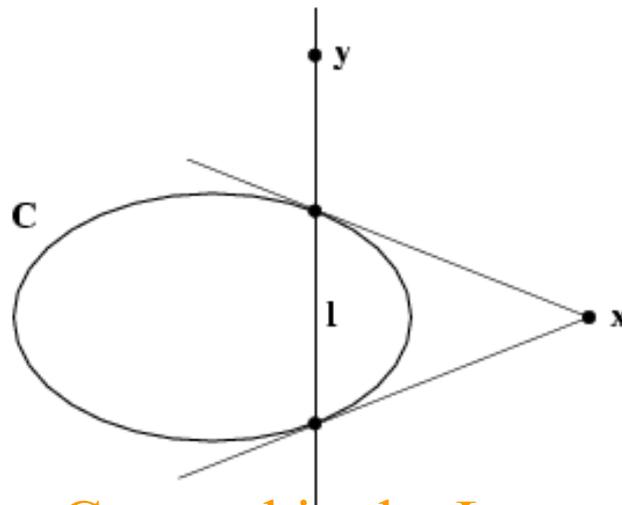
The polar plane $\Pi_{\Gamma} = QC$ of the point C with respect to the quadric Q intersects the quadric in a conic Γ . The lines tangent to Q at these points intersect at C .



*Additional Material Not Covered in the Lecture

Reminder: Pole-Polar Relationship for lines and points with respect to conics

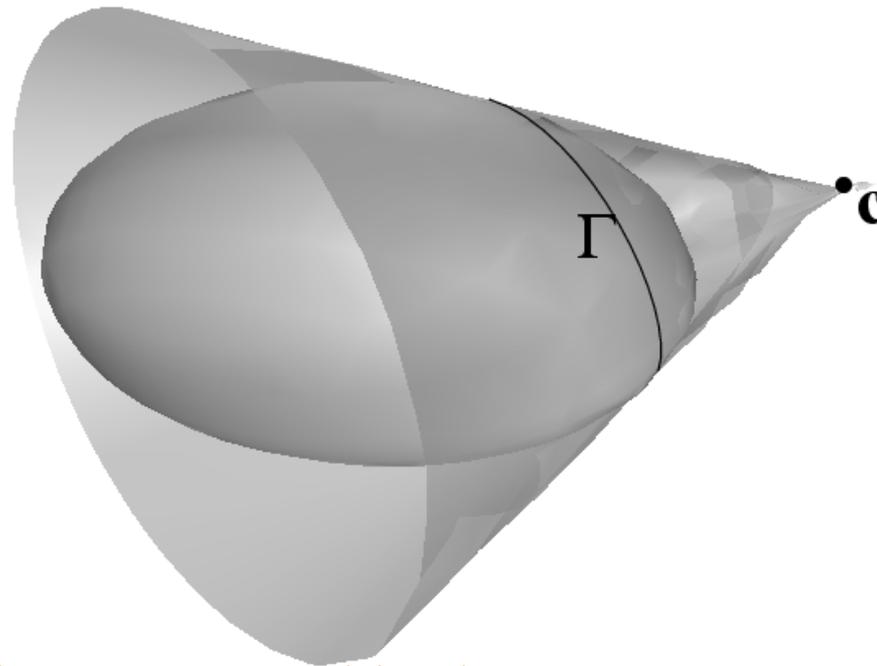
The polar line $l = Cx$ of the point x with respect to the conic C intersects the conic in two points. The two lines tangent to C at these points intersect at x



*Additional Material Not Covered in the Lecture

Pole-Polar Relationship for Planes and points with respect to quadrics

The polar plane $\Pi_{\Gamma} = QC$ of the point C with respect to the quadric Q intersects the quadric in a conic Γ . The planes tangent to Q at the points on this conic intersect at C .



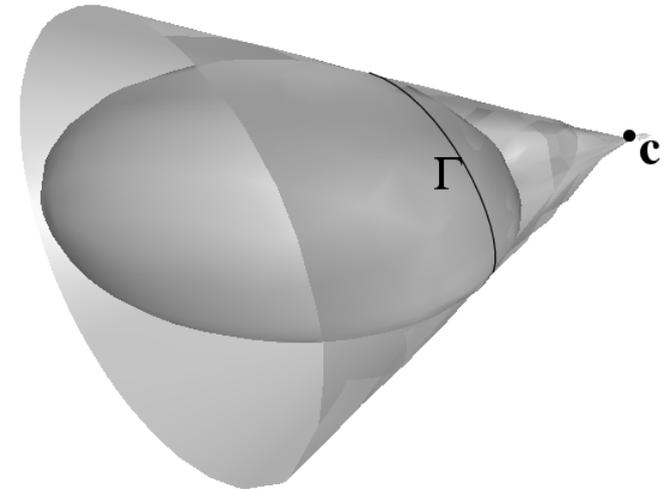
*Additional Material Not Covered in the Lecture

Action of Projective Camera on Quadrics

Forward Projection of Quadrics

$$C^* = PQ^*P^T$$

$$\Pi^T Q^* \Pi = 1^T P Q^* P^T 1 = 1^T C^* 1 = 0$$



*Additional Material Not Covered in the Lecture

Action of projective camera on quadrics

- Forward projection of quadrics:

lines tangent to the conic C satisfy:

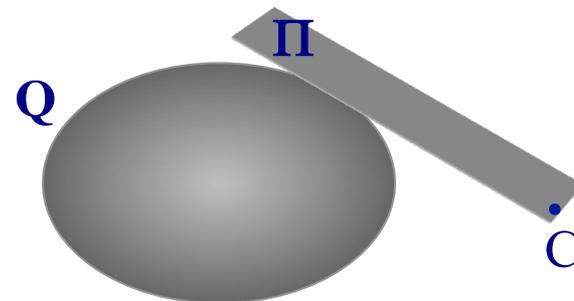
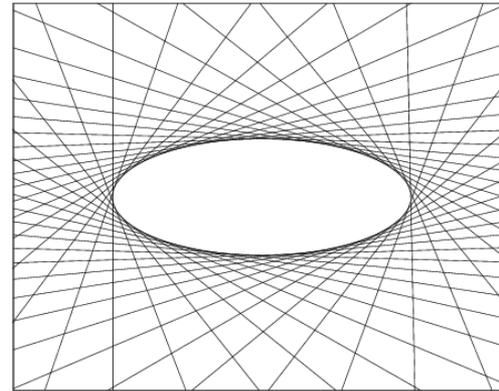
$$1^T C^* 1 = 0$$

These lines are back-projected to planes Π tangents to quadric Q , satisfying:

$$\Pi^T Q^* \Pi = 0$$

Therefore: $1^T P Q^* P^T 1 = 0$

$$C^* = P Q^* P^T$$



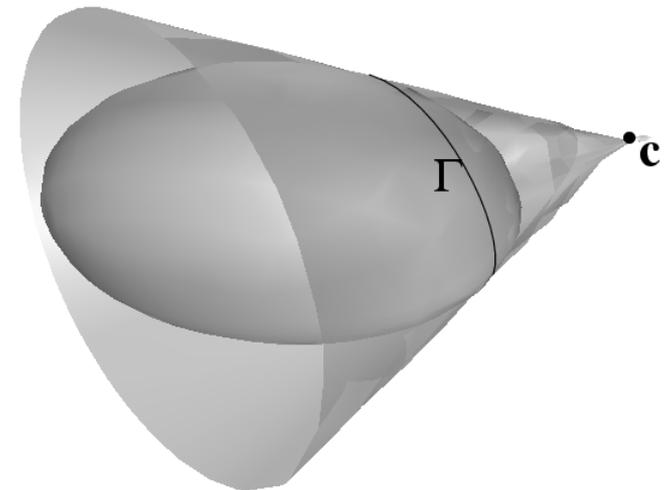
*Additional Material Not Covered in the Lecture

Action of Projective Camera on Quadrics

Forward Projection of Quadrics

$$C^* = PQ^*P^T$$

$$\Pi^T Q^* \Pi = 1^T P Q^* P^T 1 = 1^T C^* 1 = 0$$



The plane of Γ for a quadric Q and camera center C is given by $\Pi = QC$ (follows from pole-polar relation)

The cone with vertex C and tangent to the quadric Q is

$$Q_{Co} = (C^T Q C) Q - (Q C)(Q C)^T \quad Q_{Co} C = 0$$

*Additional Material Not Covered in the Lecture

Action of projective camera on quadrics

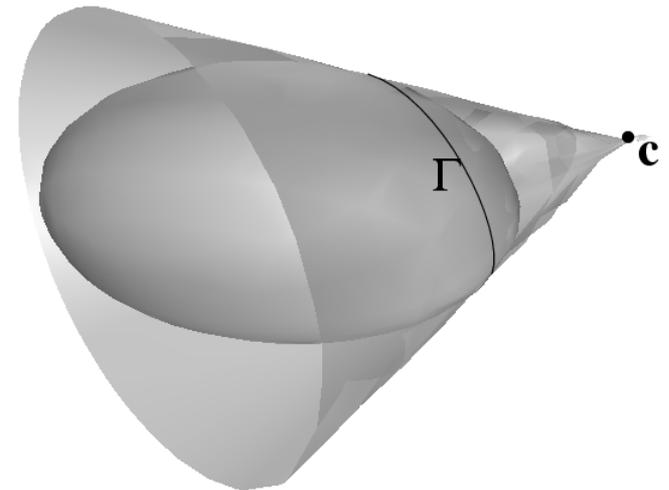
- Back-projection to cone: Example
 - The cone with vertex $C=(0,0,0,1)^T$ and tangent to the quadric Q :

$$Q = \begin{bmatrix} Q_{3 \times 3} & q \\ q^T & q_{44} \end{bmatrix}$$

$$Q_{co} = (C^T Q C) Q - (Q C)(Q C)^T$$

$$Q_{co} = \begin{bmatrix} q_{44} Q_{3 \times 3} - q q^T & 0 \\ 0 & 0 \end{bmatrix}$$

$$Q_{co} C = 0$$



*Additional Material Not Covered in the Lecture