

Exercises in 3D Computer Vision I

Exercise 1 **Epipolar Geometry and Properties of the Fundamental Matrix**

The epipolar geometry is the intrinsic projective geometry between two views I and I' . It depends only on the cameras' intrinsic parameters and their relative pose (rotation and translation between the camera centers). The **Fundamental Matrix** \mathbf{F} is a 3×3 matrix with $\text{rank}(\mathbf{F}) = 2$.

- How is the fundamental matrix \mathbf{F} related to pairs of corresponding points $\{\mathbf{x}_i \leftrightarrow \mathbf{x}'_i\}$ in the two images?
- How are the fundamental matrices \mathbf{F} , going from I to I' , and \mathbf{F}' , going from I' to I , related?
- What is the geometric meaning of the epipoles \mathbf{e} and \mathbf{e}' ? How are they related to the fundamental matrix (algebraically)?
- Are the epipoles always visible in the two views?
- How can one determine the epipolar line l' passing through a given point \mathbf{x}' ?
- What is the effect of applying the fundamental matrix \mathbf{F} to a point \mathbf{x} ?

Exercise 2 **Essential and Fundamental Matrix**

In this exercise we will derive the essential and fundamental matrices for a pair of cameras related by a rigid transformation $[\mathbf{R}|\mathbf{t}]$, by looking at a common 3D point \mathcal{P} .

Let $\mathbf{X}_1 = [X_1 \ Y_1 \ Z_1]^\top \in \mathbb{R}^3$ be the coordinates of \mathcal{P} defined in the first camera coordinate system with origin in the first camera center C_1 . Let \mathbf{R} be the rotation matrix and \mathbf{t} be the translation vector that transform the first camera coordinate system into the second camera coordinate system with origin C_2 . Remember that \mathbf{R} is a 3×3 orthogonal matrix with $\mathbf{R}\mathbf{R}^\top = \mathbf{I}$ and $\mathbf{t} \in \mathbb{R}^3$. Finally, let the coordinates of the 3D point \mathcal{P} in the second camera coordinate system be $\mathbf{X}_2 = [X_2 \ Y_2 \ Z_2]^\top \in \mathbb{R}^3$.

- How do you compute the projection $\mathbf{m}_1 = [x_1 \ y_1 \ 1]^\top$ of \mathcal{P} in the first coordinate system using the coordinates of \mathbf{X}_1 and how do you compute the projection $\mathbf{m}_2 = [x_2 \ y_2 \ 1]^\top$ of \mathcal{P} in the second coordinate system using the coordinates of \mathbf{X}_2 ?
- Give the relationship between \mathbf{X}_1 and \mathbf{X}_2 using \mathbf{R} and \mathbf{t} .
- How are the projections \mathbf{m}_1 and \mathbf{m}_2 related (always taking the first image plane as coordinate system)?
- Suppose that \mathcal{P} lies on a plane $\mathcal{P} \in \pi$, where π has a normal vector \mathbf{n}_1 (expressed in the first camera coordinate system) and its distance to C_1 is d_1 . Assuming that \mathbf{n}_1 verifies:
$$\|\mathbf{n}_1\| = \sqrt{\mathbf{n}_1^\top \mathbf{n}_1} = 1$$
, give the equation of the plane π .

e) Use the results above, including $\mathcal{P} \in \pi$, to find a transformation of the form:

$$\mathbf{m}_2 = \mathbf{T}\mathbf{m}_1$$

Hint: start from c) and use the plane condition to be able to factorize \mathbf{m}_1 on the right hand side. What type of transformation is this?

- f) Given the normalized image coordinates \mathbf{m}_1 and \mathbf{m}_2 , express the transformation of two points \mathbf{p}_1 and \mathbf{p}_2 in pixel coordinates with the help of the intrinsic matrix \mathbf{K} .
- g) The assumption that the 3D point lies on a known plane enabled us to derive a homography relating the points in the two images that correspond to \mathcal{P} . Now, try to find a relationship between \mathbf{m}_1 and \mathbf{m}_2 by omitting this assumption again. Instead, start from b) and use the plane formed by the baseline of the two cameras and the ray projecting \mathcal{P} onto the second camera. The resulting matrix is the **essential** matrix (Longuet-Higgins, Nature 1981).
- h) Introducing the knowledge about the intrinsic matrix \mathbf{K} once again, give the relation between points \mathbf{p}_1 and \mathbf{p}_2 in pixel coordinates. The new matrix we obtain here is called the **fundamental** matrix (Faugeras, ECCV 1992, Hartley ECCV 1992, Luong, PhD thesis 1992).
- i) Given the formulas derived in the previous two exercises, state the relationship between the fundamental matrix \mathbf{F} and the essential matrix \mathbf{E} .
- j) How many degrees of freedom do the following entities have?
- A general homography
 - The essential matrix
 - The fundamental matrix
- k) How can we compute \mathbf{F} ?

Exercise 3 (H) Epipolar Lines

Given a pair of images (mag_table_0.jpg and mag_table_1.jpg) and a fundamental matrix relating them \mathbf{F} (Fmat.mat), write an m-file that loads the two images and use the provided fundamental matrix \mathbf{F} to compute and display the corresponding epipolar lines for selected points. Select and display 5 points in the first image and display the corresponding epipolar lines the second image. Then do the same for the second image. Validate visually that the epipolar lines go through the selected points in the other image. Hint: use the MATLAB function `ginput` to get the point coordinates when clicking on an image.

Exercise 4 (H) Essential, Fundamental and Projection Matrices

Write an m_file that receives as input the intrinsic \mathbf{K}_i and extrinsic \mathbf{R}_i \mathbf{t}_i parameters of three pinhole cameras $i \in \{1,2,3\}$, as well as the 3D coordinates of a point in the space \mathbf{X} . Use the following to test your code:

$$\mathbf{K}_1 = \begin{bmatrix} 100 & 0 & 10 \\ 0 & 100 & 10 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{R}_1 = \begin{bmatrix} 0.9900 & 0.0993 & -0.0998 \\ -0.0894 & 0.9910 & 0.0993 \\ 0.1088 & -0.0894 & 0.9900 \end{bmatrix} \quad \mathbf{t}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{K}_2 = \begin{bmatrix} 100 & 0 & 10 \\ 0 & 100 & 10 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{R}_2 = \begin{bmatrix} 0.9900 & -0.0993 & 0.0998 \\ 0.1093 & 0.9890 & -0.0993 \\ -0.0889 & 0.1093 & 0.9900 \end{bmatrix} \quad \mathbf{t}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\mathbf{K}_3 = \begin{bmatrix} 100 & 0 & 10 \\ 0 & 100 & 10 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{R}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{t}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{X} = [1, 1, 10]$$

Note that, in order to calculate the fundamental matrix, it is advantageous to align the coordinate system to one of the camera coordinate systems (in this case all the transformations are given w.r.t. the coordinate system of the third camera).

- a) Find the projection of \mathbf{X} in the 3 images, denote them by \mathbf{x}_1 , \mathbf{x}_2 and \mathbf{x}_3 respectively. Display each projection separately for each image assuming that the image size is 20×20 .
- b) Calculate the fundamental matrices relating images I_1 and I_3 and images I_2 and I_3 .
- c) Find the epipolar line \mathbf{l}_{31} , that \mathbf{x}_1 imposes in image 3. Display the line in I_3 .
- d) Find the epipolar line \mathbf{l}_{32} , that \mathbf{x}_2 imposes in image 3. Display the line in I_3 .
- e) Find the crossing of \mathbf{l}_{31} and \mathbf{l}_{32} , and compare it with \mathbf{x}_3 . Display simultaneously in I_3 , lines \mathbf{l}_{31} and \mathbf{l}_{32} , the crossing point of both lines (use option 'x'), and the point \mathbf{x}_3 (use option 'o'). Explain.